HARVARD ENGINEERING SCHOOL

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COLLEGE ALGEBRA

BY

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Boston

ALLYN AND BACON

1889

University Press:
JOHN WILSON AND SON, CAMBRIDGE.

PREFACE.

THIS work originated in the author's desire for a course in Algebra suited to the needs of his own pupils. The increasing claims of new sciences to a place in the college curriculum render necessary a careful selection of matter and the most direct methods in the old. The author's aim has been to present each subject as concisely as a clear and rigorous treatment would allow.

The First Part embraces an outline of those fundamental principles of the science that are usually required for admission to a college or scientific school. The subjects of Equivalent Equations and Equivalent Systems of Equations are presented more fully than others. Until these subjects are more scientifically understood by the average student, it will be found profitable to review at least this portion of the First Part.

In the Second Part a full discussion of the Theory of Limits followed by one of its most important applications, Differentiation, leads to clear and concise

proofs of the Binomial Theorem, Logarithmic Series, and Exponential Series, as particular cases of Maclaurin's Formula. It also affords the student an easy introduction to the concepts and methods of the higher mathematics.

Each chapter is as nearly as possible complete in itself, so that the order of their succession can be varied at the discretion of the teacher; and it is recommended that Summation of Series, Continued Fractions, and the sections marked by an asterisk be reserved for a second reading.

In writing these pages the author has consulted especially the works of Laurent, Bertrand, Serret, Chrystal, Hall and Knight, Todhunter, and Burnside and Panton. From these sources many of the problems and examples have been obtained.

The author is indebted to Professors O. Root of Hamilton College and G. D. Olds of Rochester University, for valuable suggestions.

J. M. TAYLOR.

HAMILTON, N. Y., 1889.

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CHAPTER XXI.

ALGEBRA.

FIRST PART.

CHAPTER I.

DEFINITIONS AND NOTATION.

1. Quantity is anything that can be increased, diminished, or measured; as any portion of time or space, any distance, force, or weight.

2. To **measure** a quantity is to find how many times it contains some other quantity of the same kind taken as a *unit*, or standard of comparison.

Thus, to measure a distance, we find how many times it contains some other distance taken as a unit. To measure a portion of time, we find how many times it contains some other portion of time taken as a unit.

3. The measure of one quantity in terms of another as a unit is an **Arithmetical Number.** Hence any quantity is represented either exactly or approximately by some number.

If the unit of quantity is given, the number is concrete. Thus, 6 ft., 8 hrs., 5 lbs., are concrete numbers. If no unit of quantity is given, the number is abstract. Thus, 4, 8, 7, are abstract numbers.

4. Positive and **Negative Quantities.** Two quantities of the same kind are *opposite* in *quality*, if when united, any amount of the one annuls or destroys an equal amount of the other. Of two opposites one is said to be **Positive** in quality, and the other **Negative.**

Thus, credits and debits are opposites, since equal amounts of the two destroy each other. If we call credits positive, debits will be negative. Two forces acting along the same line in opposite directions are opposites; if we call one positive, the other is negative.

5. Algebraic Number. The sign $+$, read *positive*, and $-$, read *negative*, are used with numbers, or their symbols, to denote their *quality*, or the quality of the quantities which they represent.

Thus, if we call credit positive, $+\$5$ denotes $\$5$ of credit, and $-\$4$ denotes $\$4$ of debt. If $+8$ in. denotes 8 in. to the right, -9 in. denotes 9 in. to the left. Such numbers as $+\$5$, $-\$4$, $+8$ in., -9 in., are the concrete numbers of algebra. Each is not only the measure, but has the quality, of one of two opposites. If we disregard the units of quantity, $\$1$ and 1 inch, we obtain $+5$, -4, $+8$, -9, which are the abstract numbers of algebra.

Hence an **Algebraic Number** is the measure, and has the quality, of a positive or a negative quantity.

The *Arithmetical* or *Absolute Value* of any algebraic number is the number of units which it represents. Thus, the arithmetical value of $+4$ or -4 is 4.

The element of quality in algebraic number doubles the range of number.

Thus, the integers of arithmetic make up the simple series,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots,\ \infty; \qquad (1)$$

while the integers of algebra make up the double series,

$$-\infty, \ldots, -4, -3, -2, -1, \pm 0, +1, +2, +3, +4, \ldots, +\infty. \quad (2)$$

An algebraic number is said to be *increased* by adding a positive number, and *decreased* by adding a negative number.

If in series (2) we add $+1$ to any number, we obtain the next right-hand number. Thus, if to $+3$ we add $+1$, we obtain $+4$; if to -4 we add $+1$, we obtain -3; and so on. That is, positive numbers increase from zero, while negative numbers decrease from zero.

Hence *positive* numbers are algebraically the *greater*, the greater their arithmetical values; while *negative* numbers are algebraically the *less*, the greater their arithmetical values.

All numbers are quantities, and the term *quantity* is often used to denote number.

6. Symbols of Number. Arithmetical numbers are usually denoted by figures. Algebraic numbers are denoted by letters, or by figures with the signs $+$ and $-$ prefixed to denote their quality. A letter

usually represents both the arithmetical value and the quality of an algebraic number. Thus a may denote $+5$, -5, -8, $+17$, or any other algebraic number. When no sign is written before a symbol of number, the sign $+$ is understood.

Known Numbers, or those whose values are known, or supposed to be known, are denoted by figures, or the first letters of the alphabet, as a, b, c, a', b', c', a_1, b_1, c_1.

Unknown Numbers, or those whose values are to be found, are usually denoted by the last letters of the alphabet, as, x, y, z, x', y', z', x_1, y_1, z_1.

Quantities represented by letters are called *literal;* those represented by figures are called *numerical.*

7. Signs of Operation. The signs, $+$ (read *plus*), $-$ (read *minus*), $\times$ (read *multiplied by*), $\div$ (read *divided by*), are used in algebra to denote algebraic addition, subtraction, multiplication, and division, respectively. The use of the signs $+$ and $-$ to indicate *operations* must be carefully distinguished from their use to denote *quality.* In the literal notation, multiplication is usually denoted by writing the multiplier before the multiplicand. Thus, $ab = b \times a$. Sometimes a period is used; thus, $5 \cdot 4 = 4 \times 5$. Algebraic division is often denoted by a vinculum; thus $\frac{a}{b} = a \div b$.

8. Signs of Relation and Abbreviation. The sign of equality is $=$. The sign of identity is $\equiv$. The sign

of inequality is $>$ or $<$, the opening being toward the greater quantity.

The signs of aggregation are the parentheses ( ), the brackets [ ], the brace { }, the vinculum $\overline{\quad}$, and the bar |. They are used to indicate that two or more parts of an expression are to be taken as a whole. Thus, to indicate the product of x multiplied by $c - d$, we may write $(c-d)x$, $[c-d]x$, $\{c-d\}x$, $\overline{c-d}\,x$, or $\begin{array}{c|c} c & x \\ -d & \end{array}$.

The sign $\therefore$ is read *hence*, or *therefore;* the sign $\because$ is read *since*, or *because.*

The sign of continuation is three or more dots ..., or dashes ---, either of which is read *and so on.*

9. The result obtained by multiplying together two or more numbers is called a **Product.** Each of the numbers which multiplied together form a product, is called a **Factor** of the product.

10. A **Power** of a number is the product obtained by taking that number a certain number of times as a factor. If n is a positive integer, a^n denotes $aaaa \ldots$ to n factors, or the nth power of a. In a^n, n denotes the number of equal factors in the power, or the **Degree** of the power, and is called an **Exponent.**

11. A **Root** of a quantity is one of the equal factors into which it may be resolved.

The mth root of a is denoted by $\sqrt[m]{a}$. In $\sqrt[m]{a}$, m denotes the number of equal factors into which

a is to be resolved, and is called the **Index** of the root. The sign $\sqrt{\ }$ (a modification of r, the first letter of the word *radix*) denotes a root. If no index is written, 2 is understood.

12. Any combination of algebraic symbols which represents a number is called an **Algebraic Expression.**

13. When an algebraic expression consists of two or more parts connected by the signs $+$ or $-$, each part is called a **Term.** Thus, the expression

$$a^2 + (c - x)y + bz^2 + c \div d$$

consists of four terms.

A **Monomial** is an algebraic expression of one term; a **Polynomial** is one of two or more terms. A polynomial of two terms is called a **Binomial**; one of three terms a **Trinomial.**

14. The **Degree** of a term is the number of its literal factors. But we often speak of the degree of a term with regard to any one of its letters. Thus, $8a^2b^3x^4$, which is of the ninth degree, is of the second degree in a, the third in b, and the fourth in x.

The degree of a polynomial is that of the term of the highest degree. An expression is *homogeneous* when all its terms are of the same degree.

A *Linear* expression is one of the first degree; a *Quadratic* expression is one of the second degree.

15. Any algebraic expression that depends upon any number, as x, for its value is said to be a **Function** of x. Thus, $5x^3$ is a function of x; $5x^2 + a^3 - 7x$ is a function of both x and a; but if we wish to consider it especially with reference to x, we may call it a function of x simply.

A **Rational Integral Function** of x is one that can be put in the form

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + F,$$

in which n is a whole number, and $A, B, \ldots, F$ denote any expressions not containing x.

Thus, $ax^3 - 4x^2 - bx + c$ and $x^3 - \frac{1}{2}x$ are rational integral functions of x of the third degree.

16. The **Reciprocal** of a number is one divided by that number.

17. If a term be resolved into two factors, the first is the **Coefficient** of the second. The coefficient may be either *numerical* or *literal*. Thus, in $4abc^2$, 4 is the coefficient of abc^2, $4a$ of bc^2, and $4ab$ of c^2. When no numerical coefficient is written, 1 is understood; thus, $a = (+1)a$, and $-a = (-1)a$.

18. **Like** or **Similar Terms** are such as differ only in their coefficients. Thus, $4abc^2$ and $10abc^2$ are like terms; $6a^2b^3y^2$ and $4ab^3y^2$ are like, if we regard $6a^2$ and $4a$ as their coefficients, but unlike if 6 and 4 be taken as their coefficients.

19. A **Theorem** is a proposition to be proved.

20. A **Problem** is something to be done.

21. The **Solution** of a problem is the process by which we do what is required.

22. If A and B denote any equal algebraic expressions, then $A = B$ is called an **Equality.** A and B are called the **Members** of the equality; A the *first* member, and B the *second.*

An equality that contains only figures, or one that is true for all values of its letters, is called an **Identity.**

An equality that is true only for certain values, or sets of values, of its unknown quantities is called an **Equation.**

In writing identities, the sign $\equiv$, read *is identical with,* is often used instead of the sign $=$. The word *equation* is often used as synonymous with equality.

Thus, $5 + 4 = 9$ and $x^2 - a^2 = (x + a)(x - a)$ are identities, and may be written $5 + 4 \equiv 9$, and $x^2 - a^2 \equiv (x + a)(x - a)$. The equalities $3x - 7 = 5 - x$ and $ax = b$ are equations; the first is true only when $x = 3$, the second only when $x = b \div a$.

23. Axioms. An *axiom* is a self-evident truth. The axioms most frequently used in algebra are the following:

(i.) Quantities which are equal to the same or to equal quantities are equal to each other.

(ii.) If equal quantities be added to, or subtracted from, equal quantities, the results are equal.

(iii.) If equal quantities be multiplied or divided by the same or equal quantities, the results are equal.

(iv.) Like roots or powers of equal quantities are equal.

(v.) If the same quantity be added to and subtracted from another, the value of the latter will not be changed.

(vi.) If a quantity be both multiplied and divided by another, its value will not be changed.

24. Algebra is the science of algebraic number and the equation. It differs from arithmetic

(i.) In its number.

The number of algebra has *quality* as well as arithmetical value. The double series of numbers in algebra gives a wider range to operations. Thus, in algebra to subtract a greater quantity from a less is as natural as the reverse, while in arithmetic it is impossible.

(ii.) In its symbols of number.

Arithmetical symbols of number represent particular values; while in algebra any value may in general be attributed to the letters employed. Thus, arithmetic is confined to operations upon particular numbers, while algebra is adapted to the investigation of general principles. Again, in arithmetic all the different numbers which enter a problem are blended

together in the result, so that no trace is left of the operations by which the solution was effected. But in an algebraic solution all the literal numbers remain distinct, so that the result is a formula showing how all the data of the problem have been combined in the solution.

(iii.) In its method of solving problems.

In arithmetic we use identities, but very seldom equations; and a problem is solved by analyzing it. In algebra the characteristic instrument is the equation. To solve a problem, we translate its conditions into equations and solve these equations. The algebraic method renders easy the solution of many problems of which the arithmetical solution would be very difficult, or impossible.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

ADDITION.

25. The **Algebraic Sum** of two numbers with the same sign is the sum of their arithmetical values with their common sign prefixed; the **Algebraic Sum** of two numbers with opposite signs is the difference of their arithmetical values preceded by the sign of the arithmetically greater. Thus, the sum of -4 and -7 is -11; that of $-4b$ and $+5b$ is $+b$; that of -8 and $+4$ is -4. To indicate a sum we use the sign $+$. Thus, to indicate the sum of $-4a$, $+7b$, and $-3c$, we write

$$(-4a) + (+7b) + (-3c), \qquad (1)$$

or

$$-4a + 7b - 3c. \qquad (2)$$

Expression (2), denoting the same thing as (1), is properly called a sum.

26. Algebraic Addition is the operation of finding the simplest expression for the algebraic sum of two or more numbers. Hence by addition like terms are united into one term, while the addition of unlike terms can only be indicated.

27. Law of Commutation. *The value of a sum is not changed by changing the order of its parts;* that is,

$$a + b - c = a - c + b = b - c + a.$$

28. Law of Association. *The sum is the same, however its parts are grouped;* that is,

$$a + b - c = (a + b) - c = a + (b - c).$$

29. From the laws of §§ 27 and 28, we have the following rule for the addition of polynomials:

Write the polynomials under each other, so that like terms shall be in the same column; then add the columns separately, beginning at the left.

SUBTRACTION.

30. Having given an algebraic sum and one of its parts, **Algebraic Subtraction** is the operation of finding the other part.

The given sum is called the **Minuend**, the given part the **Subtrahend**, and the required part the **Remainder**, or **Difference.**

31. *To find the difference between two quantities, add to the minuend the subtrahend with its quality changed.*

Let A and B denote any two algebraic expressions, then $A + B$ will be their sum. Now if to the sum $A + B$ we add one part with its quality changed, as $-B$, we obtain the other part, A.

32. COROLLARY. *To change the quality of a polynomial, we change the sign of each of its terms.*

For changing the sign of each term of a polynomial evidently does not change its arithmetical value, but does change its quality.

Hence *parentheses preceded by the sign — may be removed if the sign of each of the included terms be changed.*

Thus, $ac - (m - 2cn + 3ax) = ac - m + 2cn - 3ax$. If parentheses are found in bracketed expressions, remove the brackets first.

REMARK. In arithmetic addition involves increase, and subtraction decrease; but in algebra addition may involve decrease, and subtraction increase.

MULTIPLICATION.

33. To **multiply** one algebraic number by another is to find the product of their arithmetical values and prefix the proper sign.

34. Let $+1$ as a factor denote to add once; then -1 as a factor will denote to subtract once.

Hence $+1$ into a quantity $=$ the quantity itself;
and -1 " " $= -$ " " "

35. Law of Signs. *Two like signs produce $+$; two unlike, $-$.*

For

$$(+3)(+4) = (+1)3(+4) = (+1)(+12) = +12. \quad \S\,34.$$
$$(-3)(-4) = (-1)3(-4) = (-1)(-12) = +12. \quad \S\,34.$$
$$(+3)(-4) = (+1)3(-4) = (+1)(-12) = -12.$$
$$(-3)(+4) = (-1)3(+4) = (-1)(+12) = -12.$$

In this proof the signs are all signs of quality.

36. COROLLARY 1. Since $a - b = (+1)a + (-1)b$, the sign before each term in any polynomial may be regarded as the sign of quality of the numerical coefficient of that term; hence the law of signs in § 35 holds in multiplying one polynomial by another.

37. COROLLARY 2. *Any product containing an odd number of negative factors will be* $-$; *all others*, $+$.

Hence changing the quality of an even number of factors will not change the product; but changing the quality of an odd number of factors will change the quality of the product.

38. Law of Commutation. Any change in the order of factors will evidently not change the arithmetical value of their product; and from the law of signs it will not change the quality of their product.

Hence *the value of a product is not changed by changing the order of its factors;* that is, $abcd = dcab = bcad$.

39. Law of Association. Any change in the grouping of factors will evidently not change either the arithmetical value or quality of their product.

Hence *a product is the same, however its factors are grouped;* that is, $a\,b\,c\,d\,e = (a\,b)\,(c\,d\,e) = (a\,b\,c)\,(d\,e)$.

40. COROLLARY. Since $a\,(b\,c\,d) = a\,b\,(c\,d)$, a product is multiplied by any number a by multiplying one of its factors by a.

41. The Distributive Law. *The product of two quantities equals the sum of the products of either one into the parts of the other;* that is,

$$m\,(a + b - c + d) = m\,a + m\,b - m\,c + m\,d.$$

For

$$\begin{aligned} m\,a + m\,b - m\,c + m\,d &= a\,m + b\,m - c\,m + d\,m & \S\ 38. \\ &= (a + b - c + d)\,m & \S\ 25. \\ &= m\,(a + b - c + d). & \S\ 38. \end{aligned}$$

42. Law of Exponents. If m and n are positive integers, by definition we have

$$\begin{aligned} a^m\,a^n &= (a\,a\,a \ldots \text{to } m \text{ factors})\,(a\,a\,a \ldots \text{to } n \text{ factors}) & \\ &= a\,a\,a \ldots \text{to } m + n \text{ factors} & \S\ 39. \\ &= a^{m+n}. & \S\ 10. \end{aligned}$$

43. From the commutative and distributive laws of multiplication we have the three following rules:

(i.) To multiply monomials together, *multiply together their numerical coefficients, observing the law of signs; after this result write the product of the literal factors, observing the law of exponents* (§ 38).

(ii.) To multiply a polynomial by a monomial, *multiply each term of the polynomial by the monomial, and add the results* (§ 41).

In applying the law of signs, each term must be considered as having the sign which precedes it.

(iii.) To multiply one polynomial by another, *multiply the multiplicand by each term of the multiplier, and add the results thus obtained* (§ 41).

44. Let the student state in words the following important theorems:

$$(a + c)^2 = a^2 + 2\,ac + c^2. \qquad (1)$$

$$(a - c)^2 = a^2 - 2\,ac + c^2. \qquad (2)$$

$$(a + c)(a - c) = a^2 - c^2. \qquad (3)$$

DIVISION.

45. Having given an algebraic product and one factor, **Algebraic Division** is the operation of finding the other factor. The given product is called the **Dividend**, the given factor the **Divisor**, and the required factor the **Quotient**.

46. Law of Signs. *Like signs produce* +; *unlike*, −. Let d = divisor, q = quotient; then $q\,d$ = dividend. Now, by § 35, if d and $q\,d$ have like signs, q must be +; while if d and $q\,d$ have unlike signs, q must be −.

Hence changing the quality of both dividend and divisor does not affect the quotient; but changing the quality of either the dividend or the divisor changes the quality of the quotient.

47. $$A \times \frac{1}{m} = \frac{A}{m}. \qquad (1)$$

For multiplying each member of (1) by m we obtain

$$A = A.$$

By (1), $$\frac{abc}{e} = (ab)\,c\,\frac{1}{e} = ab\frac{c}{e}. \qquad \S\ 39.$$

That is, *any product may be divided by any number by dividing one of its factors by that number.*

48. Let D = the dividend, d = the divisor, and q = the quotient; then $D = dq$.

Hence, by §§ 23, 40, 47, we have

$$mD = d\,(mq), \text{ or } \frac{D}{m} = d\,\frac{q}{m}; \qquad (1)$$

$$D = (md)\,\frac{q}{m}, \text{ or } D = \frac{d}{m}\,(mq); \qquad (2)$$

and $$mD = (md)\,q, \text{ or } \frac{D}{m} = \left(\frac{d}{m}\right) q. \qquad (3)$$

From equations (1), (2), and (3), respectively, it follows that,

(i.) *Multiplying or dividing the dividend by any quantity multiplies or divides the quotient by the same quantity.*

(ii.) *Multiplying or dividing the divisor by any quantity divides or multiplies the quotient by the same quantity.*

(iii.) *Multiplying or dividing both dividend and divisor by the same quantity does not affect the quotient.*

49. Law of Exponents. If m and n are positive integers, and $m > n$,

$$\frac{a^m}{a^n} = \frac{a\,a\,a\,a \ldots \text{to } m \text{ factors}}{a\,a\,a\,a \ldots \text{to } n \text{ factors}}$$

$$= a\,a\,a \ldots \text{to } m - n \text{ factors} \qquad \text{§ 48.}$$

$$= a^{m-n}. \qquad (1)$$

50. COROLLARY. If in (1) $m = n$, the first member is evidently 1, and the second is a^0; hence $a^0 = 1$.

That is, *any quantity with zero as an exponent equals unity.*

51. The Distributive Law. *A quotient equals the sum of the quotients of the parts of the dividend divided by the divisor.*

For $$\frac{a + c - d}{b} = (a + c - d)\frac{1}{b} \qquad \text{§ 47.}$$

$$= a\frac{1}{b} + c\frac{1}{b}\,d - \frac{1}{b} \qquad \text{§ 41.}$$

$$= \frac{a}{b} + \frac{c}{b} - \frac{d}{b}. \qquad \text{§ 47.}$$

52. *To divide one monomial by another.*

(i.) If all the literal factors of the divisor appear in the dividend, *divide the numerical coefficient of the dividend by that of the divisor, observing the law of signs; then divide the literal parts, observing the law of exponents* (§§ 47, 48).

(ii.) If all the literal factors of the divisor do not appear in the dividend, *cancel all the factors common to both the dividend and divisor* (§§ 47, 48).

53. From the distributive law we have the following two rules:

(i.) To divide a polynomial by a monomial, *divide each term of the polynomial by the monomial and add the results.*

(ii.) To divide one polynomial by another, *arrange both dividend and divisor according to the powers of some letter. Find the first term of the quotient by dividing the first term of the dividend by the first term of the divisor. Multiply the divisor by the term thus found, and subtract the product from the dividend. Treat this remainder as a new dividend and repeat the process until there is no remainder, or until a remainder is found which will not contain the divisor. Write the remainder over the divisor as a part of the quotient.*

The several products and the remainder are the parts into which the process has separated the dividend; and the quotient found is made up of the quotients of these parts divided separately by the divisor. Hence by the distributive law it is the quotient required.

54. Detached Coefficients. If two polynomials involve but one letter, or are homogeneous and involve but two letters, much labor is saved in finding their product or quotient by writing simply their coefficients. The coefficient of any missing term is zero, and must be written in order with the others.

(1) Multiply $3x^3 + 2x^2 - 8$ by $x^2 + 3 - 5x$.

$$
\begin{array}{rrrrrr}
3 & +2 & +0 & -8 & & \\
1 & -5 & +3 & & & \\
\hline
3 & +2 & +0 & -8 & & \\
 & -15 & -10 & -0 & +40 & \\
 & & +9 & +6 & +0 & -24 \\
\hline
3 & -13 & -1 & -2 & +40 & -24
\end{array}
$$

Hence the product is $3x^5 - 13x^4 - x^3 - 2x^2 + 40x - 24$.

(2) Divide $2x^3 - 8x + x^4 + 12 - 7x^2$ by $x^2 + 2 - 3x$.

$$
\begin{array}{rrrrr|l}
1 & +2 & -7 & -8 & +12 & 1-3+2 \\
\cline{6-6}
1 & -3 & +2 & & & 1+5+6 \\
\cline{1-3}
 & +5 & -9 & -8 & & \\
 & +5 & -15 & +10 & & \\
\cline{2-4}
 & & +6 & -18 & +12 & \\
 & & +6 & -18 & +12 & \\
\cline{3-5}
\end{array}
$$

Hence the quotient is $x^2 + 5x + 6$.

EXERCISE I.

1. Find by multiplication the value of $(x+y)^3$, $(x+y)^4$, $(x+y)^5$, $(x+y)^6$, $(x+y)^7$, $(x+y)^8$.

Verify by division or multiplication, and fix in mind, the following identities:

2. If n is any positive whole number,

$$\frac{x^n - y^n}{x - y} \equiv x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}.$$

If $x = 1$, this identity becomes,

$$\frac{1 - y^n}{1 - y} \equiv 1 + y + y^2 + y^3 + \cdots + y^{n-2} + y^{n-1}.$$

3. If n is any positive even number,

$$\frac{x^n - y^n}{x + y} \equiv x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + xy^{n-2} - y^{n-1}.$$

4. If n is any positive odd number,

$$\frac{x^n + y^n}{x + y} \equiv x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1}.$$

5. Show that $x^n + y^n$ is not exactly divisible by $x + y$ or $x - y$, when n is any even whole number.

CHAPTER III.

FRACTIONS.

55. An **Algebraic Fraction** is the indicated quotient of one number divided by another. The dividend is called the **Numerator**, and the divisor the **Denominator** of the fraction. The numerator and denominator of a fraction are called its terms. In fractions division is denoted by the vinculum.

Thus, $\frac{a-b}{c+d}$ denotes the quotient of $a-b$ divided by $c+d$. Here the vinculum between the terms serves as a sign both of aggregation and of division.

56. **Law of Signs.** The law of signs in fractions is the same as that in division. The sign before a fraction is the sign of its coefficient.

Thus, $$-\frac{-a}{b} = (-1)\frac{-a}{b} = \frac{a}{b}. \qquad §§ 37, 46.$$

57. An **Entire** or **Integral Number** is one which has no fractional part.

A **Mixed Number** is one which has both an entire and a fractional part.

58. The terms *Simple Fraction*, *Complex Fraction*, *Compound Fraction*, and *Common Denominator* are defined in algebra as in arithmetic.

59. The **Lowest Common Denominator** of two or more literal fractions is the expression of lowest degree that is exactly divisible by the denominators of each of the fractions.

60. To reduce a fraction to an equivalent entire or mixed number, *perform, in whole or in part, the indicated operation of division.*

61. To reduce a mixed number to an equivalent fraction, *multiply the entire part by the denominator, to the product add the numerator, and under the sum write the denominator* (§ § 23, 51).

62. To reduce a fraction to its lowest terms, *cancel all the factors common to both the numerator and the denominator* (§ 48).

63. To reduce fractions to a common denominator, *multiply both terms of each fraction by the denominators of all the other fractions. Or, find the lowest common denominator of the given fractions. Then multiply both terms of each fraction by the quotient of the lowest common denominator divided by the denominator of that fraction.*

This operation will not change the value of the fraction (§ 48), and the resulting fractions will evidently have a common denominator.

64. To add or subtract fractions, *reduce them to a common denominator, add or subtract their numera-*

tors, and place the sum or remainder over the common denominator (§§ 48, 51).

65. To multiply a fraction by an entire number, *multiply the numerator or divide the denominator by the entire number* (§ 48).

66. To divide a fraction by an entire number, *divide the numerator or multiply the denominator by the entire number* (§ 48).

67. $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{b} \times \frac{c \div c}{d} \qquad \text{§§ 23, 65, 66.}$$

$$= \frac{ac}{bd} \times \frac{c \div c}{d \div d} = \frac{ac}{bd}. \qquad \text{§§ 23, 65, 66.}$$

Hence *the product of two or more fractions equals the product of their numerators divided by the product of their denominators.*

68. $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{b} \div \frac{c}{d \div d} \qquad \text{§§ 48, 65.}$$

$$= \frac{ad}{bc} \div \frac{c \div c}{d \div d} = \frac{ad}{bc}. \qquad \text{§§ 48, 65.}$$

Hence *the quotient of one fraction divided by another equals the product of the first multiplied by the second inverted.*

69. COROLLARY. Since $\frac{1}{1} \div \frac{a}{b} = \frac{b}{a}$, the reciprocal of a fraction equals the fraction inverted.

CHAPTER IV.

THEORY OF EXPONENTS.

70. If a, b, m, n, denote any numbers, the five laws of exponents may be expressed as follows:

$$a^m \times a^n = a^{m+n}. \qquad (1)$$

$$\frac{a^m}{a^n} = a^{m-n}. \qquad (2)$$

$$(a^m)^n = a^{mn}. \qquad (3)$$

$$(ab)^m = a^m b^m. \qquad (4)$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}. \qquad (5)$$

These laws hold for any exponents, whether they be integral, fractional, positive, or negative.

71. *To prove the five laws, when the exponents* m *and* n *are positive integers.*

(i.) Law (1) is proved in § 42.

(ii.) Law (2) is proved in § 49, when $m > n$ or $m = n$.

(iii.) $(a^m)^n = a^m a^m \ldots$ to n factors § 10.

$= a^{m+m+\ldots \text{ to } n \text{ terms}}$ Law (1)

$= a^{mn}$.

(iv.) $(ab)^m = ab \cdot ab \cdot ab \ldots$ to m factors § 10.

$= (aa \ldots$ to m factors$)(bb \ldots$ to m factors$)$

$= a^m b^m.$ §§ 38, 39.

(v.) $\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots$ to m factors § 10.

$= \frac{aaa \cdots \text{ to } m \text{ factors}}{bbb \cdots \text{ to } m \text{ factors}}$ § 67.

$= \frac{a^m}{b^m}.$

72. A **Positive Fractional Exponent** denotes a root of a power. The denominator indicates the root, and the numerator the power; that is, $a^{\frac{r}{s}} = \sqrt[s]{a^r}$.

73. Let r and s be any positive whole numbers,

and let $\sqrt[s]{a} = c$, or $a = c^s$;

then $(\sqrt[s]{a})^r = c^r$,

and $a^r = (c^s)^r = c^{rs} = (c^r)^s$.

$\therefore \sqrt[s]{a^r} = c^r = (\sqrt[s]{a})^r$.

Hence $a^{\frac{r}{s}}$ *denotes either* $\sqrt[s]{a^r}$ *or its equal* $(\sqrt[s]{a})^r$.

74. Negative Exponents. If we assume law (2), § 70, to hold when $m = 0$, we have

$$\frac{1}{a^n} = a^{-n}.$$

That is, a^{-n} *denotes the reciprocal of* a^n.

75. *To prove the five laws, when the exponents* m *and* n *are positive fractions.*

(i.) Let $p, q, r, s,$ denote any positive integers; then by § 73 we have

$$a^{\frac{p}{q}}a^{\frac{r}{s}} = (a^{\frac{1}{q}}a^{\frac{1}{q}}\cdots \text{to } p \text{ factors})\,(a^{\frac{1}{s}}a^{\frac{1}{s}}\cdots \text{to } r \text{ factors}),$$

and $$a^{\frac{p}{q}+\frac{r}{s}} = (a^{\frac{1}{q}}a^{\frac{1}{q}}\cdots \text{to } p \text{ factors})\,(a^{\frac{1}{s}}a^{\frac{1}{s}}\cdots \text{to } r \text{ factors}).$$

$$\therefore\ a^{\frac{p}{q}}a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

(ii.) $$\frac{a^{\frac{p}{q}}}{a^{\frac{r}{s}}} = a^{\frac{p}{q}} \times \frac{1}{a^{\frac{r}{s}}}$$

$$= a^{\frac{p}{q}}a^{-\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

(iii.) $$\left(a^{\frac{p}{q}}\right)^r = a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{to } r \text{ factors}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\ldots \text{to } r \text{ terms}}$$

$$= a^{\frac{rp}{q}}.$$

$$\therefore\ \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \left(a^{\frac{rp}{q}}\right)^{\frac{1}{s}} \qquad \text{§§ 23, 72.}$$

$$= \left[\left(a^{rp}\right)^{\frac{1}{q}}\right]^{\frac{1}{s}}. \qquad \text{§ 72.}$$

Now one of the s equal factors of one of the q equal factors of any number is evidently one of the qs equal factors of that number; that is,

$$\left[\left(a^{rp}\right)^{\frac{1}{q}}\right]^{\frac{1}{s}} = \left(a^{rp}\right)^{\frac{1}{qs}}.$$

$$\therefore\ \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \left(a^{rp}\right)^{\frac{1}{qs}} = a^{\frac{rp}{qs}}. \qquad (1)$$

(iv.) $$(ab)^{\frac{1}{s}} = \left(a^{\frac{s}{s}}b^{\frac{s}{s}}\right)^{\frac{1}{s}}$$

$$= \left(a^{\frac{1}{s}}b^{\frac{1}{s}} \cdot a^{\frac{1}{s}}b^{\frac{1}{s}} \cdots \text{to } s \text{ factors}\right)^{\frac{1}{s}} \qquad \text{§§ 38, 39.}$$

$$= a^{\frac{1}{s}}b^{\frac{1}{s}}.$$

$$\therefore\ (ab)^{\frac{r}{s}} = \left(a^{\frac{1}{s}} b^{\frac{1}{s}}\right)^r$$

$$= \left(a^{\frac{1}{s}}\right)^r \left(b^{\frac{1}{s}}\right)^r$$

$$= a^{\frac{r}{s}} b^{\frac{r}{s}}. \qquad \S\ 73.$$

(v.) Let $\dfrac{a}{b} = c$, or $a = bc$;

then $\left(\dfrac{a}{b}\right)^{\frac{r}{s}} = c^{\frac{r}{s}}$,

and $a^{\frac{r}{s}} = (bc)^{\frac{r}{s}} = b^{\frac{r}{s}} c^{\frac{r}{s}}$, or $\dfrac{a^{\frac{r}{s}}}{b^{\frac{r}{s}}} = c^{\frac{r}{s}}$.

$$\therefore\ \left(\frac{a}{b}\right)^{\frac{r}{s}} = \frac{a^{\frac{r}{s}}}{b^{\frac{r}{s}}}.$$

COROLLARY. By (1), $a^{\frac{p}{q}} = \left(a^{\frac{p}{q}}\right)^{\frac{r}{r}} = a^{\frac{rp}{rq}}$.

76. *To prove the five laws, when the exponents are negative.*

Let h and k be any positive numbers.

(i.) $a^{-h} a^{-k} = \dfrac{1}{a^h} \times \dfrac{1}{a^k}$ § 74.

$= \dfrac{1}{a^{h+k}}$ §§ 67, 71.

$= a^{-(h+k)} = a^{-h-k}$. § 74.

(ii.) $\dfrac{a^{-h}}{a^{-k}} = \dfrac{a^{-h} a^k}{a^{-k} a^k} = a^{-h} a^k$ § 48.

$= a^{-h+k}$.

(iii.) $(a^{-h})^{-k} = 1 \div \left(\frac{1}{a^h}\right)^k$ § 74.

$= 1 \div \frac{1}{a^{hk}} = a^{hk}.$ §§ 71, 68.

(iv.) $(ab)^{-k} = \frac{1}{(ab)^k}$ § 74.

$= \frac{1}{a^k b^k} = \frac{1}{a^k} \cdot \frac{1}{b^k} = a^{-k} b^{-k}.$

(v.) $\left(\frac{a}{b}\right)^{-k} = 1 \div \left(\frac{a}{b}\right)^k$ § 74.

$= 1 \div \frac{a^k}{b^k} = \frac{b^k}{a^k} \cdot \frac{a^{-k}}{a^{-k}} \cdot \frac{b^{-k}}{b^{-k}} = \frac{a^{-k}}{b^{-k}}.$

NOTE. The introduction of fractional and negative exponents is evidently not necessary; but they supply us with a new notation of very great convenience.

77. If we use the term *power* to signify what is indicated by any exponent, the five laws of exponents may be stated as follows:

(i.) *The product of the* m*th and the* n*th power of any number equals the* (m + n)*th power of that number.*

(ii.) *The quotient of the* m*th power of any number divided by its* n*th power equals the* (m − n)*th power of that number.*

(iii.) *The* n*th power of the* m*th power of any number equals the* m n*th power of that number.*

(iv.) *The* m*th power of the product of any number of factors equals the product of the* m*th powers of those factors.*

COROLLARY. The rth root of the product of two or more factors equals the product of their rth roots.

(v.) *The* m*th power of the quotient of any two quantities equals the quotient of their* m*th powers.*

COROLLARY. The rth root of the quotient of any two numbers equals the quotient of their rth roots.

78. To affect a monomial product with a given exponent, *multiply the exponent of each factor by the given exponent.*

This rule follows from laws (3) and (4).

$$\text{Thus,}\quad \left(4\,a^2\,b^{-\frac{1}{2}}\,c^{\frac{3}{5}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}}\left(a^2\right)^{\frac{3}{2}}\left(b^{-\frac{1}{2}}\right)^{\frac{3}{2}}\left(c^{\frac{3}{5}}\right)^{\frac{3}{2}}$$

$$= 8\,a^3\,b^{-\frac{3}{4}}\,c^{\frac{9}{10}}.$$

79. $$\frac{a^k}{b^{-k}} = \frac{a^k}{b^{-k}}\cdot\frac{b^k}{b^k}\cdot\frac{a^{-k}}{a^{-k}} = \frac{b^k}{a^{-k}}.$$

Hence *a factor may be changed from one term of a fraction to the other if the sign of its exponent be changed.*

EXERCISE 2.

1. Multiply $3\,a^{\frac{2}{3}}\,b^{\frac{5}{6}}\,c^{\frac{7}{4}}$ by $2\,a^{\frac{1}{3}}\,b^{\frac{3}{4}}\,c^3$; $7\,a^{\frac{7}{2}}\,x^{-n}\,y^{-\frac{5}{4}}$ by $6\,a^2\,x^{m-n}\,y^{\frac{5}{4}}$.

2. Perform the operations indicated by the exponents in each of the following expressions: $\left(2\,a^{\frac{1}{2}}\,x^{-\frac{1}{8}}\,y^{\frac{1}{6}}\right)^4$; $\left(125\,a^{\frac{3}{4}}\,x^{-\frac{4}{3}}\right)^{\frac{2}{3}}$; $(8\,a^6\,b^3\,c^9\,d^{-3})^{-\frac{1}{3}}$; $\left(64\,a^{-\frac{3}{2}}\,x^{-\frac{3}{5}}\right)^{\frac{4}{3}}$; $\left(a^{\frac{2}{5}}\,b^{-\frac{2}{7}}\,c^{-\frac{3}{4}}\right)^{-\frac{4}{3}}$; $\left(a^{-m}\,b^{-\frac{r}{s}}\,x^{-\frac{y}{c}}\,c^{\frac{r}{s}}\right)^{-\frac{p}{q}}$.

3. In each of the following expressions introduce the coefficient within the parentheses: $8\,(a^2 - x^2)^{\frac{3}{2}}$; $a^5\,(a + a^2 x)^{\frac{5}{2}}$; $x^5\,(1 - x^2)^{\frac{3}{4}}$; $x^3\,(a - x)^{\frac{5}{6}}$; $x^2\,(x^2 - a\,y)^{-\frac{2}{3}}$.

$$(8)^1 = (8)^{\frac{2}{3}\cdot\frac{3}{2}} = \left(8^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}}.$$

$$\therefore\ 8\,(a^2 - x^2)^{\frac{3}{2}} = 4^{\frac{3}{2}}\,(a^2 - x^2)^{\frac{3}{2}} = (4\,a^2 - 4\,x^2)^{\frac{3}{2}}.$$

4. Simplify $\dfrac{\left(1 + \dfrac{y^2}{x^2}\right)^{\frac{5}{2}}}{x^2}$; $\dfrac{\left(1 + \dfrac{y^3}{x^3}\right)^{\frac{4}{3}}}{x}$; $\dfrac{\left(1 - \dfrac{x^4}{y^4}\right)^{\frac{5}{4}}}{x^3}$.

$$\frac{\left(1 + \dfrac{y^2}{x^2}\right)^{\frac{5}{2}}}{x^2} = \frac{\left(\dfrac{x^2 + y^2}{x^2}\right)^{\frac{5}{2}}}{x^2} = \frac{\dfrac{(x^2+y^2)^{\frac{5}{2}}}{(x^2)^{\frac{5}{2}}}}{x^2} = \frac{\dfrac{(x^2+y^2)^{\frac{5}{2}}}{x^5}}{x^2} = \frac{(x^2+y^2)^{\frac{5}{2}}}{x^7}.$$

5. Remove a monomial factor from within the parentheses in each of the following expressions: $3\,(a^2 - a^3\,b^2)^{\frac{3}{2}}$; $2\,(9\,a^2\,b - 18\,a\,b^3)^{\frac{5}{2}}$; $\frac{2}{3}\,(27\,a^4\,b^7 - 54\,a^3\,b^4)^{\frac{4}{3}}$.

6. Simplify $\sqrt[3]{\dfrac{a^6\,b^{-3}\,y^3}{x^{-3}\,c^6\,z^9}}$; $\sqrt[5]{\dfrac{a^{10}\,b^{-\frac{5}{4}}\,c^2}{z^{-5}\,y^5\,h^{\frac{10}{3}}}}$; $\sqrt[r]{\dfrac{a^n\,x^s\,c^{-2r}\,y^{-\frac{2r}{p}}}{y^{\frac{2r}{s}}\,z^r\,h^{-\frac{2}{3}r}}}$.

7. Square the following binomials: $b^n x^{-m} - a^d x^{\frac{2}{n}}$; $y^{-n} x^{m-r} c^x - d^r n^{-3}$; $r^{-c} z^{n-x} c + 5\, l^n b^{-a}$.

8. Free of negative exponents

$$\frac{a^{-2} b^3}{y^{-2} x^3};\quad \frac{5 x^{-\frac{3}{2}} y^{-\frac{1}{2}}}{7 m^{-2} y^{-5} z^{-\frac{3}{4}}};\quad \frac{9 x^{-\frac{5}{6}} y^{-\frac{3}{4}} z^{-1}}{7 a^{-\frac{7}{4}} b^{-3} c^{-n}}.$$

9. Simplify $\dfrac{\dfrac{3x}{2} + \dfrac{x-1}{3}}{\dfrac{13}{6}(x+1) - \dfrac{x}{3} - \dfrac{5}{2}}$; $\dfrac{\dfrac{x^2+y^2}{y} - x}{\dfrac{1}{y} - \dfrac{1}{x}} \times \dfrac{x^2-y^2}{x^3+y^3}$.

10. Simplify $\dfrac{1}{1 - \dfrac{1}{1 + \dfrac{1}{x}}}$; $\dfrac{1}{1 + \dfrac{x}{1 + x + \dfrac{2x^2}{1-x}}}$.

11. Simplify $\dfrac{\dfrac{a+b}{c+d} + \dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d} + \dfrac{a-b}{c+d}}$; $\dfrac{\dfrac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 - y^3}}{\dfrac{(a+x)^2}{x^2 + xy + y^2}}$.

12. Simplify $\left(\dfrac{x+2y}{x+y} + \dfrac{x}{y}\right) \div \left(\dfrac{x+2y}{y} - \dfrac{x}{x+y}\right)$.

13. Simplify $\left(\dfrac{x^2}{y^3} + \dfrac{1}{x}\right) \div \left(\dfrac{x}{y^2} - \dfrac{1}{y} + \dfrac{1}{x}\right)$.

CHAPTER V.

FACTORING, HIGHEST COMMON DIVISOR, LOWEST COMMON MULTIPLE.

80. Factoring is the operation of finding the factors of a given product. It is the converse of multiplication.

81. When each term of an expression contains the same factor, the expression is divisible by that factor.

Thus, $x^2y + xy^2 + 4ax - 3bx = x(xy + y^2 + 4a - 3b)$.

Also, $ac(ac + d) + b(ac + d) = (ac + b)(ac + d)$.

BINOMIALS.

82. Whatever be the values of m and n,

$$x^{2m} - y^{2n} = (x^m + y^n)(x^m - y^n).$$

That is, *the difference between any two quantities is equal to the product of the sum and difference of their square roots.*

Thus, $a^8 - x^6 = (a^4 + x^3)(a^4 - x^3)$.

83. From the examples of Exercise 1, page 21, we have,

(i.) $x^n - y^n$ is divisible by $x - y$ if n is *any* whole number; and the n terms in the quotient are *all positive.*

(ii.) $x^n - y^n$ is divisible by $x + y$ if n is even; and the n terms in the quotient are *alternately positive and negative.*

Thus, $$a^{\frac{4}{5}} - b^{\frac{4}{5}} = \left(a^{\frac{1}{5}}\right)^4 + \left(b^{\frac{1}{5}}\right)^4$$
$$= \left(a^{\frac{1}{5}} + b^{\frac{1}{5}}\right)\left(a - a^{\frac{2}{5}} b^{\frac{1}{5}} + a^{\frac{1}{5}} b^{\frac{2}{5}} - b^{\frac{3}{5}}\right).$$

(iii.) $x^n + y^n$ is divisible by $x + y$ if n is odd; and the n terms in the quotient are *alternately positive and negative.*

Thus, $$x^{10} + y^5 = (x^2 + y)(x^8 - x^6 y + x^4 y^2 - x^2 y^3 + y^4).$$

(iv.) $x^n + y^n$ is not divisible by $x + y$ or $x - y$ when n is even.

84. For any value of n we have,

$$x^{4n} + y^{4n} = x^{4n} + y^{4n} + 2x^{2n}y^{2n} - 2x^{2n}y^{2n}$$
$$= (x^{2n} + y^{2n})^2 - (x^n y^n \sqrt{2})^2$$
$$= (x^{2n} + y^{2n} + x^n y^n \sqrt{2})(x^{2n} + y^{2n} - x^n y^n \sqrt{2}). \quad (1)$$

For $n = 1$, (1) becomes

$$x^4 + y^4 = (x^2 + y^2 + xy\sqrt{2})(x^2 + y^2 - xy\sqrt{2}).$$

TRINOMIALS.

85. $$x^2 \pm 2ax + a^2 = (x \pm a)^2.$$

That is, *if two terms of a trinomial are positive, and the third is $\pm$ twice their square roots, the trinomial equals the square of the sum or difference of the two square roots.*

86. $$x^2 + (a + b)x + ab = (x + a)(x + b);$$

hence $$x^2 + cx + d = (x + a)(x + b),$$

if $$a + b = c \text{ and } ab = d. \qquad (1)$$

Equations (1) can always be solved for a and b by the method of § 166; hence a trinomial of the form $x^2 + cx + d$ can be resolved into two linear factors in x. Equations (1) however may often be solved by inspection.

EXAMPLE. Factor $x^6y^4 - 11x^3y^2 + 30$.

$$x^6y^4 - 11x^3y^2 + 30 = (x^3y^2)^2 - 11(x^3y^2) + 30;$$

hence $$a + b = -11, \text{ and } ab = 30;$$

therefore $$a = -5,\ b = -6;$$

whence $$x^6y^4 - 11x^3y^2 + 30 = (x^3y^2 - 5)(x^3y^2 - 6).$$

87. $$nx^2 + cx + d = \frac{(nx)^2 + c(nx) + nd}{n}.$$

Now $(nx)^2 + c(nx) + nd$ can be factored by § 86.

Hence any trinomial of the form $nx^2 + cx + d$ can be resolved into two linear factors in x.

EXAMPLE. Factor $15x^2 - 7x - 2$.

$$15x^2 - 7x - 2 = \frac{(15x)^2 - 7(15x) - 30}{15}$$
$$= \frac{(15x - 10)(15x + 3)}{5 \cdot 3} = (3x - 2)(5x + 1).$$

88. When, by increasing one of its terms, a trinomial can be made a perfect square, it can be factored by § 82.

EXAMPLE. Factor $x^4 - 3a^4x^2 + 9a^8$.

$$x^4 - 3a^4x^2 + 9a^8 = x^4 + 6a^4x^2 + 9a^8 - 9a^4x^2$$
$$= (x^2 + 3a^4)^2 - (3a^2x)^2$$
$$= (x^2 + 3a^4 + 3a^2x)(x^2 + 3a^4 - 3a^2x).$$

89. A polynomial of four or more terms may often be factored by properly arranging its terms, and applying the foregoing principles.

1. $cx^2 - cy^2 - ax^2 + ay^2 = c(x^2 - y^2) - a(x^2 - y^2)$
$= (c - a)(x - y)(x + y).$

2. $c^2a^2 + 4ab^2c + 4b^4 - 16f^4 = (ca + 2b^2)^2 - (4f^2)^2$
$= (ca + 2b^2 + 4f^2)(ca + 2b^2 - 4f^2).$

3. $x^4 - x^2 - 9 - 2a^2x^2 + a^4 + 6x = (x^2 - a^2)^2 - (x - 3)^2$
$= (x^2 - a^2 + x - 3)(x^2 - a^2 - x + 3).$

EXERCISE 3.

Resolve into their simplest factors:

1. $a^2c^2 + acd + abc + bd$.

2. $a^2y^3 - b^2yx^2 - a^2dy^2 + b^2dx^2$.

3. $10x^3 + 30x^2y - 8xy^2 - 24y^3$.

4. $(a+b)^4 - 1$.

5. $a^2b^2 - 3abc - 10c^2$.

6. $98 - 7x - x^2$.

7. $x^6 + 1$.

8. $x^8 - 1$.

9. $7 + 10x + 3x^2$.

10. $6x^2 + 7x - 3$.

11. $2x^2 - 5xy + 3y^2$.

12. $12x^2 - 23xy + 5y^2$.

13. $9x^2 + 24xy + 16y^2$.

14. $x^4 + 16x^2 + 256$.

15. $81a^4 + 9a^2b^2 + b^4$.

16. $2 + 7x - 15x^2$.

17. $3x^2 + 41x + 26$.

18. $31x - 35 - 6x^2$.

19. $a^4 + b^4 - c^4 - d^4 + 2a^2b^2 - 2c^2d^2$.

20. $1 - a^2x^2 - b^2y^2 + 2abxy$.

21. $a^2x - b^2x + a^2y - b^2y$.

22. Resolve $x^4y - x^2y^3 - x^3y^2 + xy^4$ into four factors.

23. Resolve $a^9 - 64a^3 - a^6 + 64$ into six factors.

24. Resolve $4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$ into four factors.

25. Write out the following quotients:

$$(x^{\frac{5}{3}} - y^{\frac{5}{4}}) \div (x^{\frac{1}{3}} - y^{\frac{1}{4}});$$

$$(x^{\frac{4}{3}} - y^{\frac{4}{5}}) \div (x^{\frac{1}{3}} + y^{\frac{1}{5}});$$

$$(x^{\frac{5}{2}} + a^{\frac{5}{3}}) \div (x^{\frac{1}{2}} + a^{\frac{1}{3}}).$$

HIGHEST COMMON DIVISOR.

90. A **Common Divisor** of two or more expressions is an expression that divides each of them exactly. Two expressions are *prime* to each other if they have no common factor other than unity.

91. The **Highest Common Divisor** of two or more algebraic expressions is the expression of highest degree that will divide each of them exactly.

The abbreviation H. C. D. is often used for the words *highest common divisor*.

92. When the given expressions can be resolved into their simple factors, or such as are prime to each other, their H. C. D. is obtained by taking the product of all their common factors, each being raised to the lowest power in which it occurs in any of the expressions.

Thus, the H. C. D. of $6(x-1)(x+2)^3$ and $3(x-1)^2(x+2)^2(x-3)$ is $3(x-1)(x+2)^2$.

93. When the given expressions cannot be resolved into their factors, the method of finding their H. C. D. is based on the following theorem:

If $A = BQ + R$, then the H. C. D. of A and B is the same as the H. C. D. of B and R.

Since $R = A - BQ$, by § 81, every factor common to A and B divides R; hence every factor

common to A and B is common to B and R. Conversely, since $A = BQ + R$ every factor common to B and R divides A; hence every factor common to B and R is common to A and B.

Hence the H. C. D. of B and R is the H. C. D. of A and B.

94. *To find the H. C. D. of two algebraic quantities.*

Let A and B denote any two rational integral functions of x, whose H. C. D. is required, the degree of B not being greater than that of A.

Divide A by B and let the quotient be Q_1 and the remainder R_1. Divide B by R_1 and let the quotient be Q_2 and the remainder R_2. Divide R_1 by R_2 and let the quotient be Q_3 and the remainder R_3. Continue this process until the remainder is zero, or does not contain x. If the last remainder is zero, the last divisor is the H. C. D.; if the last remainder is not zero, there is no H. C. D.

From the process above described, it follows that

$$\begin{aligned} A &= B\,Q_1 + R_1, \\ B &= R_1\,Q_2 + R_2, \\ R_1 &= R_2\,Q_3 + R_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_{n-2} &= R_{n-1}\,Q_n + R_n. \end{aligned}$$

Now by § 93 the pairs of expressions, A and B, B and R_1, R_1 and R_2, ..., R_{n-2} and R_{n-1}, all have the same H. C. D.

(i.) If $R_n = 0$, $R_{n-2} = R_{n-1} Q_n$. Hence the H. C. D. of R_{n-1} and R_{n-2}, or $R_{n-1} Q_n$, is R_{n-1}. Hence R_{n-1} is the H. C. D. of A and B.

(ii.) If R_n is not zero, the H. C. D. of A and B is the H. C. D. of R_{n-1} and R_n (§ 93). But, since R_n does not contain x, R_{n-1} and R_n have no common factor in x. Hence A and B have no common divisor.

95. COROLLARY. To avoid fractions, and to otherwise simplify the work in finding the H. C. D., it is important to note that at any stage of the process,

(i.) We may multiply either the dividend or the divisor by any quantity that is not a factor of the other.

(ii.) We may remove from either the dividend or the divisor any factor that is not common to both.

(iii.) We may remove from both the dividend and the divisor any common factor, provided it is reserved as a factor of the H. C. D.

96. To find the H. C. D. of three expressions, A, B, C, find the H. C. D. of A and B, and then find the H. C. D. of this result and C. This last H. C. D. will be the H. C. D. of A, B, and C.

LOWEST COMMON MULTIPLE.

97. A **Common Multiple** of two or more expressions is an expression that is exactly divisible by each of them.

The **Lowest Common Multiple** (abbreviated L. C. M.) of two or more expressions is the expression of lowest degree that is exactly divisible by each of them.

98. Hence when two or more expressions can be resolved into their factors the L. C. M. of these expressions is the product of their factors, each being raised to the highest power in which it occurs in any of the expressions.

99. To find the L. C. M. of two expressions, as A and B, when they cannot be factored, *divide A by the H. C. D. of A and B and multiply the quotient by B.*

For the L. C. M. of A and B must evidently contain all the factors of B, and in addition all the factors of A not common to A and B; hence the rule.

EXERCISE 4.

Find the H. C. D. of the following expressions:

1. $x^3 + 2x^2 - 8x - 16,\ x^3 + 3x^2 - 8x - 24.$
2. $2x^4 - 2x^3 + x^2 + 3x - 6,\ 4x^4 - 2x^3 + 3x - 9.$
3. $4x^5 + 14x^4 + 20x^3 + 70x^2,\ 8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3.$

Find the L. C. M. of the following expressions:

4. $x^4 + ax^3 + a^3x + a^4,\ x^4 + a^2x^2 + a^4.$
5. $x^3 - 9x^2 + 26x - 24,\ x^3 - 12x^2 + 47x - 60.$

CHAPTER VI.

INVOLUTION, EVOLUTION, SURDS, IMAGINARIES.

100. Involution is the operation of finding a power of a number.

Evolution is the operation of finding a root of a number.

For the involution and evolution of monomials see § 78, of binomials see § 275.

101. A root is said to be even or odd according as its index is even or odd.

By the law of signs it follows that,

(i.) Any *odd* root of a quantity has the same sign as the quantity itself.

(ii.) Any *even* root of a *positive* quantity may be either positive or negative. In this chapter only positive even roots are considered.

(iii.) Any *even* root of a *negative* quantity is not found in the series of algebraic numbers thus far considered.

An even root of a negative number is called an *Imaginary* number. For sake of distinction all other numbers are called *Real*.

102. *To find the square root of any number.*

The rule is given by the formula,

$$(a + b)^2 = a^2 + (2a + b)b,$$

in which a represents the first term of the root or the part of the root already found; b the next term of the root; $2a$ the trial divisor in obtaining b; and $2a + b$ the true divisor.

In finding the square root of any polynomial, as $4x^4 + 9y^4 + 13x^2y^2 - 6xy^3 - 4x^3y$, its terms should be arranged according to the descending powers of some letter, and the work may be arranged as below:

$$
\begin{array}{rll}
 & & \left|\,\underline{2x^2 - xy + 3y^2}\right. \\
 & & 4x^4 - 4x^3y + 13x^2y^2 - 6xy^3 + 9y^4 \\
a^2 = & & 4x^4 \\
2a + b = & 4x^2 - xy & -4x^3y + 13x^2y^2 \\
(2a + b)b = & & -4x^3y + x^2y^2 \\
2a + b = & 4x^2 - 2xy + 3y^2 & 12x^2y^2 - 6xy^3 + 9y^4 \\
(2a + b)b = & & 12x^2y^2 - 6xy^3 + 9y^4
\end{array}
$$

At first $a = 2x^2$ and $b = -xy$; then $a = 2x^2 - xy$ and $b = 3y^2$.

The root is placed above the number for convenience. In extracting the square root of any number expressed in the decimal notation, we first divide it into periods of *two* figures each, beginning with units' place. We then proceed essentially as with the polynomial above, bearing in mind that a denotes tens with reference to b.

103. *To find the cube root of any number.*

The rule is given by the formula,

$$(a + b)^3 = a^3 + (3\,a^2 + 3\,a\,b + b^2)b,$$

in which a represents the first term of the root or the part of the root already found; b the next term of the root; $3\,a^2$ the trial divisor in obtaining b; and $3\,a^2 + 3\,ab + b^2$ the true divisor.

In finding the cube root of any polynomial, as $8\,x^6 - 36\,x^5 + 66\,x^4 + 1 - 63\,x^3 - 9\,x + 33\,x^2$, its terms should be arranged according to the descending powers of some letter, and the work may be arranged as below:

		$2x^2-3x+1$
		$8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$
$a^3 =$		$8x^6$
$3a^2 =$	$12x^4$	$-36x^5+66x^4-63x^3$
$3ab+b^2 =$	$-18x^3+9x^2$	
$(3a^2+3ab+b^2)b =$		$-36x^5+54x^4-27x^3$
$3a^2 =$	$12x^4-36x^3+27x^2$	$12x^4-36x^3+33x^2-9x+1$
$3ab+b^2 =$	$6x^2-9x+1$	
$(3a^2+3ab+b^2)b =$		$12x^4-36x^3+33x^2-9x+1$

At first $a = 2\,x^2$ and $b = -3\,x$; then $a = 2\,x^2 - 3\,x$ and $b = 1$.

In finding the cube root of any number expressed in the decimal notation, we first divide it into periods of *three* figures each, beginning with units' place. We then proceed essentially as with the polynomial above, a denoting tens with reference to b.

104. In finding the fourth root of any number, we may obtain the square root of its square root, or follow the rule given by the formula,

$$(a+b)^4 = a^4 + (4a^3 + 6a^2b + 4ab^2 + b^3)b.$$

The rule for finding the fifth root is given by the formula,

$$(a+b)^5 = a^5 + (5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)b.$$

The sixth root may be obtained by finding the cube root of its square root, or by using the formula,

$$(a+b)^6 = a^6 + (6a^5 + 15a^4b + 20a^3b^2 + 15a^2b^3 + 6ab^4 + b^5)b.$$

In like manner we may obtain any root of a quantity.

EXERCISE 5.

Find the square root of

1. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.
2. $9x^6 - 12x^5 + 22x^4 + x^2 + 12x + 4$.
3. 384524.01.
4. 0.24373969.

Find the cube root of

5. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
6. $24x^4y^2 + 96x^2y^4 - 6x^5y + x^6 - 96xy^5 + 64y^6 - 56x^3y^3$.
7. 3241792.
8. 191.102976.
9. Find the fifth root of
   $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.
10. Find the fourth root of
   $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

SURDS.

105. If the root of a quantity cannot be exactly obtained, its indicated root is called a **Surd** or **Irrational Quantity**. All quantities which are not surds are called *rational quantities*. The *order* of a surd is indicated by the index of the root. Thus, $\sqrt{a}$ and $\sqrt[n]{a}$ are respectively surds of the second and nth orders. The surds of most frequent occurrence are those of the second order; they are often called *quadratic surds*.

106. Surds of different orders may be transformed into others of the same order. The order may be any common multiple of the orders of the given surds; but usually it is most convenient to choose the L. C. M.

Thus, $\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$,

and $\sqrt[3]{b^2} = b^{\frac{2}{3}} = b^{\frac{4}{6}} = \sqrt[6]{b^4}$.

107. A surd is in its simplest form when the smallest possible entire quantity is under the radical sign. Surds are said to be **Like** when they have, or can be so reduced as to have, the same irrational factor; otherwise they are said to be **Unlike**.

Thus, $2\sqrt{5}$ and $\frac{1}{2}\sqrt{5}$ are like surds, so also are $\sqrt{18}$ and $\sqrt{\frac{1}{2}}$.

108. In adding or subtracting surds reduce them to their simplest form by the principles of § 70, and combine those that are like.

109. The product or quotient of surds of the same order may be obtained by the laws of exponents (§ 70). If they are of different orders they may be reduced to the same order.

Thus, $\quad x\sqrt[3]{a} \times b\sqrt[4]{c} = x\,b\,a^{\frac{1}{3}}\,c^{\frac{1}{4}}$

$$= x\,b\,a^{\frac{4}{12}}\,c^{\frac{3}{12}} = x\,b\,\sqrt[12]{a^4 c^3}.$$

110. When two binomial quadratic surds differ only in the sign of a surd term, they are said to be **Conjugate.**

Thus, $\sqrt{a} + \sqrt{b}$ is conjugate to $\sqrt{a} - \sqrt{b}$, or $-\sqrt{a} + \sqrt{b}$.

The product of two conjugate surds is evidently rational.

111. The quotient of one surd by another may be found by expressing the quotient as a fraction, and then multiplying both terms of the fraction by such a factor as will render the denominator rational. This process is called *rationalizing the denominator.* The cases that most frequently occur are the three following:

I. When the denominator is a monomial surd, as $\sqrt[n]{y}$, the rationalizing factor is evidently $y^{\frac{n-1}{n}}$.

II. When the denominator is a binomial quadratic surd, as $\sqrt{a} + \sqrt{b}$, the rationalizing factor is its conjugate, $\sqrt{a} - \sqrt{b}$ or $-\sqrt{a} + \sqrt{b}$.

III. When the denominator is of the form $\sqrt{a} + \sqrt{b} + \sqrt{c}$, first multiply both terms of the fraction by $\sqrt{a} + \sqrt{b} - \sqrt{c}$; the denominator thus becomes $(\sqrt{a} + \sqrt{b})^2 - (\sqrt{c})^2$, or $(a + b - c) + 2\sqrt{ab}$. Then multiply both terms of the fraction by $(a + b - c) - 2\sqrt{ab}$; and the denominator becomes the rational quantity $(a + b - c)^2 - 4ab$.

112. *To find a factor that will rationalize any given binomial surd, as* $\sqrt[r]{a} \pm \sqrt[s]{b}$.

Let n be the L. C. M. of r and s; then $(\sqrt[r]{a})^n$ and $(\sqrt[s]{b})^n$ are both rational and so also is their sum or difference. There are three cases

I. When the given surd is $a^{\frac{1}{r}} - b^{\frac{1}{s}}$; then by § 83

$$\frac{\left(a^{\frac{1}{r}}\right)^n - \left(b^{\frac{1}{s}}\right)^n}{a^{\frac{1}{r}} - b^{\frac{1}{s}}} \equiv a^{\frac{n-1}{r}} + a^{\frac{n-2}{r}} b^{\frac{1}{s}} + a^{\frac{n-3}{r}} b^{\frac{2}{s}} + \cdots + b^{\frac{n-1}{s}}$$

$\equiv$ the rationalizing factor.

II. When the given surd is $a^{\frac{1}{r}} + b^{\frac{1}{s}}$, and n is *even*; then by § 83

$$\frac{\left(a^{\frac{1}{r}}\right)^n - \left(b^{\frac{1}{s}}\right)^n}{a^{\frac{1}{r}} + b^{\frac{1}{s}}} \equiv a^{\frac{n-1}{r}} - a^{\frac{n-2}{r}} b^{\frac{1}{s}} + a^{\frac{n-3}{r}} b^{\frac{2}{s}} - \cdots - b^{\frac{n-1}{s}}$$

$\equiv$ the rationalizing factor.

III. When the given surd is $a^{\frac{1}{r}} + b^{\frac{1}{s}}$, and n is *odd;* then by § 83,

$$\frac{\left(a^{\frac{1}{r}}\right)^n + \left(b^{\frac{1}{s}}\right)^n}{a^{\frac{1}{r}} + b^{\frac{1}{s}}} \equiv a^{\frac{n-1}{r}} - a^{\frac{n-2}{r}}b^{\frac{1}{s}} + a^{\frac{n-3}{r}}b^{\frac{2}{s}} - \cdots + b^{\frac{n-1}{s}}$$
$$\equiv \text{the rationalizing factor.}$$

In each case the rational product is the numerator of the fraction in the first member of the identity.

EXAMPLE. Find the factor that will rationalize $\sqrt{3} + \sqrt[3]{5}$.

$$\frac{(\sqrt{3})^6 - (\sqrt[3]{5})^6}{\sqrt{3} + \sqrt[3]{5}} = 3^{\frac{5}{2}} - 3^{\frac{4}{2}}5^{\frac{1}{3}} + 3^{\frac{3}{2}}5^{\frac{2}{3}} - 3 \cdot 5 + 3^{\frac{1}{2}}5^{\frac{4}{3}} - 5^{\frac{5}{3}}$$
$$= 9\sqrt{3} - 9\sqrt[3]{5} + 3\sqrt{3}\sqrt[3]{25} - 15 + 5\sqrt{3}\sqrt[3]{5} - 5\sqrt[3]{25}.$$

EXERCISE 6.

Find the value of

1. $(\sqrt{2} + \sqrt{3} - \sqrt{5})(\sqrt{2} + \sqrt{3} + \sqrt{5})$.

2. $(4 + 3\sqrt{2}) \div (5 - 3\sqrt{2})$.

3. $17 \div (3\sqrt{7} + 2\sqrt{3})$.

4. $(2\sqrt{3} + 7\sqrt{2}) \div (5\sqrt{3} - 4\sqrt{2})$.

5. $\dfrac{\sqrt{3} + \sqrt{2}}{2 - \sqrt{3}} \div \dfrac{7 + 4\sqrt{3}}{\sqrt{3} - \sqrt{2}}$.

6. $\dfrac{2\sqrt{15} + 8}{5 + \sqrt{15}} \div \dfrac{8\sqrt{3} - 6\sqrt{5}}{5\sqrt{3} - 3\sqrt{5}}$.

Rationalize the denominator of

7. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

8. $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

9. $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

10. $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}$.

11. $\dfrac{\sqrt[3]{3}-1}{\sqrt[3]{3}+1}$.

12. $\dfrac{3\sqrt{3}}{\sqrt{3}+\sqrt[6]{9}}$.

13. $\dfrac{\sqrt{2}\,\sqrt[3]{3}}{\sqrt[3]{3}+\sqrt{2}}$.

14. $\dfrac{\sqrt{8}+\sqrt[3]{4}}{\sqrt{8}-\sqrt[3]{4}}$.

113. *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

If possible let $\sqrt{a}=n+\sqrt{m}$,

then $$a=n^2+m+2n\sqrt{m};$$

$$\therefore\ \sqrt{m}=\frac{a-n^2-m}{2n},$$

which is impossible, since a surd cannot equal a rational quantity.

114. *In any equality containing rational quantities and quadratic surds, the rational parts in the two members are equal, and also the irrational parts.*

Suppose $a + \sqrt{b} = x + \sqrt{y}.$

If possible let $a = x + m;$

then $x + m + \sqrt{b} = x + \sqrt{y};$

or $\sqrt{y} = m + \sqrt{b},$

which is impossible by § 113.

Hence $a = x,$

and therefore $\sqrt{b} = \sqrt{y}.$

115. *To find the square root of* $a \pm 2\sqrt{b}$.

Since $\sqrt{x + y \pm 2\sqrt{xy}} = \sqrt{x} \pm \sqrt{y};$

therefore $\sqrt{a \pm 2\sqrt{b}} = \sqrt{x} \pm \sqrt{y},$ (1)

if $x + y = a,$ and $xy = b.$ (2)

Solving equations (2) as simultaneous (§ 166), and substituting the results in (1), we obtain

$$\sqrt{a \pm 2\sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - 4b}}{2}}.$$

Equations (2) may often be solved by inspection.

EXAMPLE. Find the square root of $13 + 2\sqrt{30}.$

Here $x + y = 13$ and $xy = 30;$

$\therefore\ x = 10$ and $y = 3.$

Hence $\sqrt{13 + 2\sqrt{30}} = \sqrt{10} + \sqrt{3}.$

EXERCISE 7.

Find by inspection the square root of

1. $7 - 2\sqrt{10}$.
2. $5 + 2\sqrt{6}$.
3. $8 - 2\sqrt{7}$.
4. $18 - 8\sqrt{5}$.
5. $47 - 4\sqrt{33}$.
6. $15 - 4\sqrt{14}$.
7. $19 + 8\sqrt{3}$.
8. $11 + 4\sqrt{6}$.
9. $29 + 6\sqrt{22}$.

IMAGINARY QUANTITIES.

116. Imaginary quantities frequently occur in mathematical investigations, and their use leads to valuable results. By the methods of Trigonometry, any imaginary expression may readily be reduced to the form of a quadratic imaginary expression. We give below some of the laws of combination of quadratic imaginaries.

117. By the definition of a square root we have

$$\sqrt{-1} \times \sqrt{-1} = -1.$$

$$\therefore \quad \sqrt{a}\sqrt{-1} \times \sqrt{a}\sqrt{-1} = -a;$$

that is, $$(\sqrt{a}\sqrt{-1})^2 = -a = (\sqrt{-a})^2.$$

$$\therefore \quad \sqrt{-a} = \sqrt{a}\sqrt{-1}.$$

By this principle any quadratic imaginary term may evidently be reduced to the form $c\sqrt{-1}$.

Thus, $$\sqrt{-11\,a^2} = \sqrt{11\,a^2}\sqrt{-1} = a\sqrt{11}\sqrt{-1}.$$

118. To add or subtract quadratic imaginaries, reduce each imaginary term to the form $c\sqrt{-1}$, and then proceed as in the case of other surds.

Thus, $\sqrt{-4}+\sqrt{-9}=2\sqrt{-1}+3\sqrt{-1}=5\sqrt{-1}.$

119. *To find the successive powers of* $\sqrt{-1}$.

$$(\sqrt{-1})^2 = -1;$$
$$\therefore\ (\sqrt{-1})^3 = (-1)(\sqrt{-1}) = -\sqrt{-1};$$
$$\therefore\ (\sqrt{-1})^4 = (-1)(\sqrt{-1})^2 = +1;$$
$$\therefore\ (\sqrt{-1})^{4n} = (+1)^n = +1,$$

in which n is any positive integer. Hence, in general,

$$(\sqrt{-1})^{4n} = 1; \qquad (\sqrt{-1})^{4n+2} = -1.$$
$$(\sqrt{-1})^{4n+1} = \sqrt{-1}; \quad (\sqrt{-1})^{4n+3} = -\sqrt{-1}.$$

120. An expression containing both real and imaginary terms is called an **Imaginary** or **Complex** expression. The general typical form of a quadratic imaginary expression is $a+b\sqrt{-1}$. If $a=0$, this becomes $b\sqrt{-1}$.

121. Two imaginary expressions are said to be **Conjugate** when they differ only in the sign of the imaginary part.

Thus, $a-b\sqrt{-1}$ is conjugate to $a+b\sqrt{-1}$.

122. *The sum and product of two conjugate imaginary expressions are both real.*

For $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$

and $$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - (-b^2)$$
$$= a^2 + b^2$$

The positive square root of the product $a^2 + b^2$ is called the **Modulus** of each of the conjugate expressions, $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$.

123. *If two imaginary expressions are equal, the real parts must be equal and also the imaginary parts.*

For let $$a + b\sqrt{-1} = c + d\sqrt{-1};$$

then $$a - c = (d - b)\sqrt{-1}.$$

Hence $$(a - c)^2 = -(d - b)^2,$$

which is evidently impossible, except $a = c$ and $b = d$.

124. COROLLARY. If $a + b\sqrt{-1} = 0$, $a = 0$, and $b = 0$.

125. To multiply or divide one imaginary expression by another, reduce them each to the typical form, then proceed as in the multiplication or division of any other surds, obtaining the product or the quotient of the imaginary factors by § 119.

Thus, $$\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1}$$
$$= \sqrt{a}\sqrt{b}(\sqrt{-1})^2 = -\sqrt{ab}.$$

REMARK. The student should carefully note that the product of the square roots of two negative numbers is not equal to the square root of their product.

Thus, $\sqrt{-2} \times \sqrt{-8}$ does not equal $\sqrt{16}$.

126. When the divisor is imaginary the quotient may be found by expressing it as a fraction and then rationalizing the denominator.

Thus, $\dfrac{1}{3-2\sqrt{-3}} = \dfrac{3+2\sqrt{3}\sqrt{-1}}{9+12} = \dfrac{1}{7} + \dfrac{2\sqrt{3}}{21}\sqrt{-1}.$

EXERCISE 8.

Perform the following indicated operations:

1. $4\sqrt{-3} \times 2\sqrt{-2}$.
2. $(\sqrt{2} + \sqrt{-2})(\sqrt{2} - \sqrt{-2})$.
3. $(2\sqrt{-3})^5$.
4. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.
5. $\sqrt{-16} \div \sqrt{-4}$.
6. $(3\sqrt{-7} - 5\sqrt{-2})(3\sqrt{-7} + 5\sqrt{-2})$.
7. $(1 + \sqrt{-1}) \div (1 - \sqrt{-1})$.
8. $(4 + \sqrt{-2}) \div (2 - \sqrt{-2})$.
9. $\dfrac{3\sqrt{-2} - 2\sqrt{-5}}{3\sqrt{-2} + 2\sqrt{-5}}$.
10. $1 \div \dfrac{a - \sqrt{-x}}{a + \sqrt{-x}}$.
11. What is the modulus of $3 + 2\sqrt{-3}$? Of $5 - 3\sqrt{-2}$?

CHAPTER VII.

EQUATIONS.

127. An **Equation** is an equality that is true only for certain values, or sets of values, of its unknown quantities. Any such value, or set of values, is called a **Solution** of the equation. Equations are classified according to the number of their unknown quantities; thus, we have equations of one unknown quantity; of two unknown quantities; of three unknown quantities; and so on.

For example, the equality $5x = 15$ is an equation of one unknown quantity x; its single solution is $x = 3$. Again, $(x - 5)(x - 4) = 0$ is an equation; its two solutions are evidently $x = 5$ and $x = 4$. The equality $y = 2x + 3$ is an equation of two unknown quantities x and y; one of its solutions is $x = 1$, $y = 5$; another is $x = 2$, $y = 7$; another is $x = 3$, $y = 9$; and so on for an unlimited number of solutions.

128. An equation is said to be **Numerical** or **Literal** according as its known quantities are represented by figures only, or wholly or in part by letters.

129. When an equation contains only rational integral functions of its unknown quantities, its **Degree** is that of the term of highest degree in the unknown

quantities. Thus, the equations $x^3 + x^2 + 4 = 0$ and $xy^2 + xy = 5$ are each of the third degree.

A *Linear* equation is one of the first degree.

A *Quadratic* equation is one of the second degree.

A *Cubic* equation is one of the third degree.

A *Biquadratic* equation is one of the fourth degree.

Equations above the second degree are called *Higher Equations*.

EQUIVALENT EQUATIONS.

130. Two equations involving the same unknown quantities are said to be *Equivalent* when they have the same solutions; that is, when the solutions of either include all the solutions of the other.

Thus, $5x - 10a = 3x - 4a$ and $2x = 6a$ are equivalent equations; for the only solution of either is $x = 3a$. A single equation may be equivalent to two or more other equations.

Thus, $$(3x - 6a)(x^2 - 9b^2) = 0 \qquad (1)$$

is equivalent to the two equations

$$3x - 6a = 0, \qquad (2)$$

and $$x^2 - 9b^2 = 0. \qquad (3)$$

For any solution of (1) must evidently render one of the factors of its first member equal to zero, and hence must satisfy (2) or (3); and conversely any solution of (2) or (3) must satisfy (1).

131. *If the same quantity be added to both members of an equation, the resulting equation will be equivalent to the first.*

Let $$A = B \qquad (1)$$

represent any equation of one or more unknown quantities, and let m denote any quantity whatever; then by § 23

$$A + m = B + m. \qquad (2)$$

Now it is evident that (1) and (2) are each satisfied by any set of values of the unknown quantities that will render A and B equal, and only by such sets. Hence (1) and (2) are equivalent equations.

132. COROLLARY. By the principle of § 131 we may transpose any term from one member of an equation to the other by changing its sign. For this is the same thing as adding to both members the term to be transposed, with its sign changed.

133. *If both members of an equation be multiplied by the same known quantity, the resulting equation will be equivalent to the first.*

Let $$A = B \qquad (1)$$

represent any equation, and c any known quantity; then by § 23.

$$cA = cB. \qquad (2)$$

Now it is evident that (1) and (2) are each satisfied by any set of values of the unknown quantities that will render A and B equal, and only by such sets. Hence (1) and (2) are equivalent equations.

134. COROLLARY. The principle of § 133 is used in

(i.) Clearing an equation of fractions of which the denominators are known quantities.

(ii.) Changing the signs of all its terms, which is equivalent to multiplying both members by -1.

(iii.) Dividing both members by the same known quantity, which is equivalent to multiplying by its reciprocal.

Thus, if we multiply both members of the equation

$$\frac{x-4}{7}=\frac{x-10}{5}$$

by 35, we obtain the equivalent equation

$$5x-20=7x-70.$$

135. *If both members of an equation be multiplied by the same integral function of its unknown quantities, in general, new solutions will be introduced.*

Let $$A=B, \text{ or } A-B=0, \qquad (1)$$

represent any equation, and m any integral function of its unknown quantities; then by § 23

$$mA=mB, \text{ or } m(A-B)=0. \qquad (2)$$

Now (1) is satisfied only when A is equal to B; but (2) is satisfied not only when A is equal to B, but also in general when $m=0$. Hence the solutions of $m=0$ have been introduced by multiplying (1) by m.

Thus, if we multiply both members of the equation

$$x = 4, \text{ or } x - 4 = 0,$$

by $x - 2$, we introduce the solution of $x - 2 = 0$; for we obtain

$$(x - 2)(x - 4) = 0,$$

which is evidently equivalent to both $x - 4 = 0$ and $x - 2 = 0$.

Again, if we multiply both members of the equation

$$y = 2x, \text{ or } y - 2x = 0,$$

by $y - x$, we introduce the solutions of $y - x = 0$; for we obtain

$$(y - x)(y - 2x) = 0,$$

which is evidently equivalent to both $y - x = 0$, and $y - 2x = 0$.

·**136.** COROLLARY I. *If* m *is the denominator of a fraction in the equation* A = B, *then multiplying both members of* A = B *by* m *does not, in general, introduce new solutions.**

If no one of the solutions of $m = 0$ appears among those of the resulting equation, then evidently no solution has been introduced; if any of them do appear, those must be rejected which do not satisfy the given equation.

EXAMPLE. Solve $\dfrac{3}{x - 1} = 5 - x.$ (1)

Multiplying (1) by $x - 1$, $3 = (x - 1)(5 - x)$, (2)

or $x^2 - 6x + 8 = 0.$

Hence $(x - 4)(x - 2) = 0.$ (3)

Now the only solution that could be introduced by multiplying (1) by $x - 1$ is $x = 1$. But the solutions of (3) are $x = 4$ and $x = 2$; hence the solution $x = 1$ was not introduced.

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* The reason for this exception to § 135 is that in this case the solutions of $m = 0$ do not make both A and B finite. Thus, the solution of $x - 1 = 0$, or $x = 1$, does not render the first member of (1) finite.

To avoid introducing new solutions in clearing an equation of fractions:

(i.) Those which have a common denominator should be combined.

(ii.) Any factor common to the numerator and denominator of any fraction should be cancelled.

(iii.) When multiplying by a multiple of the denominators always use the L. C. M.

EXAMPLE. Solve $1 - \frac{x^2}{x-1} = \frac{1}{1-x} - 6.$ (1)

Transposing and combining, we have

$$1 - \frac{x^2-1}{x-1} = -6.$$

$$\therefore \quad 1-(x+1) = -6, \text{ or } x = 6.$$

But if we first clear (1) of fractions, we obtain

$$x - 1 - x^2 = -1 - 6x + 6,$$

or
$$(x-6)(x-1) = 0,$$

of which the roots are 6 and 1. But as $x = 1$ does not satisfy (1), the root 1 was introduced in clearing of fractions.

137. COROLLARY 2. In solving an equation of the form

$$mA = mB, \text{ or } m(A-B) = 0,$$

we should write its two equivalent equations,

$$m = 0, \text{ and } A - B = 0,$$

and solve each.

Thus, the equation $x^3 - 1 = 0$ may be written in the form

$$(x-1)(x^2+x+1)=0, \tag{1}$$

which is equivalent to the two equations

$$x - 1 = 0 \text{ and } x^2 + x + 1 = 0,$$

the solutions of which are readily found.

138. *If both members of an equation be raised to the same integral power, in general, new solutions will be introduced.*

Let the equation be $\quad A = B. \qquad (1)$

Squaring both members of (1) we obtain

$$A^2 = B^2, \text{ or } A^2 - B^2 = 0,$$

which can be written in the form

$$(A-B)(A+B)=0. \tag{2}$$

Now (2) is equivalent to the two equations

$$A - B = 0 \text{ and } A + B = 0.$$

Therefore the solutions of $A + B = 0$ were introduced by squaring both members of (1).

Hence if in solving an equation, we raise both its members to any power, we must reject those solutions of the resulting equation which do not satisfy the given equation.

EXAMPLE 1. Let the equation be

$$\sqrt{4-x} = x - 4. \qquad (1)$$

Squaring (1), $\quad 4 - x = x^2 - 8x + 16,$

or $\quad x^2 - 7x + 12 = 0.$

Hence $\quad (x-4)(x-3) = 0. \qquad (2)$

Now the solutions of (2) are evidently $x = 4$ and $x = 3$, of which $x = 3$ does not satisfy (1). Hence the solution $x = 3$ was introduced by squaring (1).

EXAMPLE 2. Let the equation be

$$y - 2 = x. \qquad (1)$$

Squaring (1), $(y-2)^2 = x^2$, or $(y-2)^2 - x^2 = 0.$

Hence $\quad (y - 2 - x)(y - 2 + x) = 0. \qquad (2)$

Now (2) is equivalent to the two equations

$$y + 2 - x = 0 \text{ and } y - 2 + x = 0.$$

Hence the solutions of $y - 2 + x = 0$ were introduced by squaring (1).

LINEAR EQUATIONS OF ONE UNKNOWN QUANTITY.

139. Any solution of an equation of one unknown quantity is called a **Root** of the equation.

140. By the preceding principles any Linear equation of one unknown quantity can be reduced to an equivalent equation of the form

$$ax = c. \qquad (1)$$

Dividing both members of (1) by a, we obtain

$$x = c \div a.$$

Hence *a linear equation has one, and only one root.*

EXERCISE 9.

Solve the following equations; that is, find their roots:

1. $\dfrac{x-8}{7}+\dfrac{x-3}{3}+\dfrac{5}{21}=0.$

2. $\dfrac{4(x+2)}{3}-\dfrac{6(x-7)}{7}=12.$

3. $\dfrac{7-5x}{1+x}=\dfrac{11-15x}{1+3x}.$

4. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-1}=\dfrac{1}{6}.$

5. $\dfrac{4(x+3)}{9}=\dfrac{8x+37}{18}-\dfrac{7x-29}{5x-12}.$

6. $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}.$

Reducing the first two fractions to a mixed form, we have

$$6+\frac{24}{x+1}+8+\frac{36}{x+3}=14+\frac{48}{x+1};$$

$$\therefore\quad \frac{36}{x+3}=\frac{24}{x+1},\text{ or }\frac{3}{x+3}=\frac{2}{x+1},\text{ etc.}$$

7. $\dfrac{x+5}{x+4}-\dfrac{x-6}{x-7}=\dfrac{x-4}{x-5}-\dfrac{x-15}{x-16}.$

8. $\dfrac{.3x-1}{.5x-.4}=\dfrac{.5+1.2x}{2x-.1}.$

9. $\sqrt{x-32}=16-\sqrt{x}.$

10. $\dfrac{5x-9}{\sqrt{5x}+3} - 1 = \dfrac{\sqrt{5x}-3}{2}$.

11. $\sqrt{x} - \sqrt{x - \sqrt{1-x}} = 1$.

In example 10, cancel the factor common to the terms of the first fraction. Multiplying both members by $\sqrt{5x}+3$ would introduce the root of the equation $\sqrt{5x}+3=0$. In example 11, neither of the two roots obtained will satisfy the given equation; which therefore has no root, and is impossible.

12. $\dfrac{ax-1}{\sqrt{ax}+1} = 4 + \dfrac{\sqrt{ax}-1}{2}$.

13. $\dfrac{3\sqrt{x}-4}{2+\sqrt{x}} = \dfrac{15+\sqrt{9x}}{40+\sqrt{x}}$.

14. $\dfrac{1}{\sqrt{a-x}+\sqrt{a}} + \dfrac{1}{\sqrt{a-x}-\sqrt{a}} = \dfrac{\sqrt{a}}{x}$.

15. $\sqrt{x} - \sqrt{x-8} = \dfrac{2}{\sqrt{x-8}}$.

Quadratic Equations of one Unknown Quantity.

141. By the preceding principles any quadratic equation in x can be reduced to an equivalent one of the form

$$a x^2 + b x + c = 0. \qquad (1)$$

In this equation a cannot be zero, for then the equation would cease to be a quadratic.

If b or c is zero, equation (1) assumes the form

$$a x^2 + c = 0, \text{ or } a x^2 + b x = 0. \qquad (2)$$

Either of equations (2) is said to be *Incomplete.* The first is called a *Pure* quadratic.

If $b = c = 0$, (1) becomes $a x^2 = 0$; $\therefore x = \pm 0$.

142. *To solve the pure quadratic* $a x^2 + c = 0$.

Solving the equation for x^2, we obtain

$$x^2 = -(c \div a).$$

$$\therefore \quad x = \pm \sqrt{-\frac{c}{a}}.$$

The two values of x will be real or imaginary, according as c and a have unlike or like signs.

Hence *a pure quadratic has two roots, arithmetically equal with opposite signs; both are real, or both are imaginary.*

143. *To solve the incomplete quadratic* $a x^2 + b x = 0$.

This equation may be put in the form

$$x(a x + b) = 0. \qquad (1)$$

Now (1) is equivalent to the two equations

$$x = 0 \text{ and } a x + b = 0,$$

whose roots are 0 and $-(b \div a)$, respectively.

144. *To solve the complete quadratic*

$$a x^2 + b x + c = 0. \qquad (1)$$

We first transform equation (1) so that its first member shall be a perfect square. To do this we transpose c, then multiply both members by $4a$, and finally add b^2 to both members. We thus obtain

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$$

or
$$(2ax + b)^2 = b^2 - 4ac.$$

$$\therefore \quad 2ax + b = \pm\sqrt{b^2 - 4ac};$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

Hence, to solve a complete quadratic, *transform the equation so that its first member shall be a perfect square, and then proceed as above; or put the equation in the form of* (1), *and then apply formula* (2).

145. Sum and Product of Roots. Representing the roots of $ax^2 + bx + c = 0$ by α and β, we have

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad (1)$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

Adding (1) and (2), we find the sum

$$\alpha + \beta = -\frac{b}{a}. \qquad (3)$$

Multiplying (1) by (2), we find the product

$$\alpha\beta = \frac{c}{a}. \qquad (4)$$

Dividing both members of $ax^2 + bx + c = 0$ by a, we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \qquad (5)$$

From (3) and (4) it follows that if a quadratic be put in the form of (5),

(i.) *The sum of its roots is equal to the coefficient of* x *with its sign changed.*

(ii.) *The product of its roots is equal to the known term.*

For example, the sum of the roots of the equation $3x^2 + 7x + 12 = 0$ is $-\frac{7}{3}$, and their product is 4.

146. COROLLARY 1. If the roots α and β are arithmetically equal and opposite in sign, the equation is a pure quadratic.

For if $-b \div a = \alpha + \beta = 0$, $b = 0$.

147. COROLLARY 2. If the roots are reciprocals of each other, $a = c$; and conversely, if $a = c$, the roots are reciprocals.

For if $c \div a = \alpha\beta = 1$, $a = c$; and conversely, if $a = c$, $\alpha\beta = 1$.

148. Character of Roots. From the values of α and β in (1) and (2) of § 145, it evidently follows that,

(i.) If $b^2 - 4ac > 0$, the roots are real and unequal.

(ii.) If $b^2 - 4ac = 0$, the roots are real and equal.

(iii.) If $b^2 - 4ac < 0$, the roots are conjugate imaginaries.

(iv.) If $b^2 - 4ac$ is a perfect square, the roots are rational; otherwise they are conjugate surds.

Thus, the roots of $3x^2 - 24x + 36 = 0$ are real and unequal; for here

$$b^2 - 4ac = (-24)^2 - 4 \times 3 \times 36 > 0.$$

Again, the roots of $3x^2 - 12x + 135 = 0$ are conjugate imaginaries; for here

$$b^2 - 4ac = (-12)^2 - 4 \times 3 \times 135 < 0.$$

149. Number of Roots. Since $\frac{b}{a} = -(\alpha + \beta)$ and $\frac{c}{a} = \alpha\beta$, we may write the general equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

in the form $x^2 - (\alpha + \beta)x + \alpha\beta = 0,$ (1)

or $(x - \alpha)(x - \beta) = 0.$ (2)

Now α and β are evidently the only values of x that will satisfy (2).

Hence *every quadratic equation has two, and only two roots.*

From either (1) or (2) we see that a quadratic equation may be formed of which the roots shall be any two given quantities.

Thus, if the roots are 5 and -3, by (2) the equation is

$$(x-5)(x+3)=0, \text{ or } x^2-2x-15=0.$$

If the roots are $2 \pm \sqrt{-3}$, their sum is 4, and their product is 7, hence by (1) the equation is

$$x^2-4x+7=0.$$

150. Resolution into Factors. Any quadratic expression of the form ax^2+bx+c can be factored by finding the roots, α and β, of the equation.

$$ax^2+bx+c=0.$$

For $$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$=a(x-\alpha)(x-\beta). \qquad \S\ 149.$$

EXAMPLE. Factor $2x^2-14x+36$.

The roots of the equation $2x^2-14x+36=0$ are

$$\tfrac{7}{2}+\tfrac{1}{2}\sqrt{-23} \text{ and } \tfrac{7}{2}-\tfrac{1}{2}\sqrt{-23};$$

hence $2x^2-14x+36 \equiv 2(x-\frac{7}{2}-\frac{1}{2}\sqrt{-23})(x-\frac{7}{2}+\frac{1}{2}\sqrt{-23})$.

151. Solution by a Formula. Any complete quadratic may be reduced to the form

$$x^2+px+q=0. \qquad (1)$$

Solving (1), we obtain

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}. \qquad (2)$$

Formula (2) affords the following simple rule for writing out the two roots of a quadratic equation in the form of (1):

The roots equal one half the coefficient of x, *with its sign changed, increased and diminished by the square root of the square of one half the coefficient of* x *diminished by the known term.*

Thus, to solve $x^2 + 3x + 11 = 0$, we have

$$\begin{aligned} x &= -\tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} - 11} \\ &= -\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{-35}. \end{aligned}$$

152. Solution by Factoring. In solving equations the student should always utilize the principles of factoring and equivalent equations.

EXAMPLE. Solve $x^6 = 1$. (1)

Transposing and factoring, (1) may be written in the form

$$(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = 0. \quad (2)$$

The roots of (2) are $1, -1$, and those of the two equations

$$x^2 + x + 1 = 0 \text{ and } x^2 - x + 1 = 0,$$

which are readily solved.

EXERCISE 10.

By § 145 determine the sum and the product of the roots of each of the four following equations. By § 148 find the character of the roots of each. Then solve each by § 151.

1. $5x^2 - 6x - 8 = 0$.
2. $x^2 + 11 = 7x$.
3. $2x = x^2 + 5$.
4. $5x^2 = 17x - 10$.

Form the equations whose roots are

5. $3, -8.$

6. $\frac{2}{3}, \frac{7}{3}.$

7. $3 \pm \sqrt{5}.$

8. $2 \pm \sqrt{-3}.$

9. $\frac{m}{n}, -\frac{n}{m}.$

10. $\frac{a+b}{a-b}, \frac{a-b}{a+b}.$

Solve the following equations:

11. $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}.$

12. $\frac{x+4}{x-4} + \frac{x-2}{x-3} = 6\frac{1}{3}.$

13. $\frac{3x-1}{4x+7} = 1 - \frac{6}{x+7}.$

14. $\frac{5x-7}{7x-5} = \frac{x-5}{2x-13}.$

15. $\frac{5}{x-2} - \frac{4}{x} = \frac{3}{x+6}.$

16. $\frac{x+a}{x-2a} + \frac{x-2a}{x+a} = 1.$

17. $\frac{3}{2(x^2-1)} - \frac{1}{4(x+1)} = \frac{1}{8}.$

18. $\frac{x+1}{x+2} + \frac{x-1}{x-2} = \frac{2x-1}{x-1}.$

19. $\frac{\sqrt{4x}+2}{4+\sqrt{x}} = \frac{4-\sqrt{x}}{\sqrt{x}}.$

20. $\frac{a+2b}{a-2b} = \frac{a^2}{(a-2b)x} - \frac{4b^2}{x^2}.$

21. $a^2x^2 - 2a^3x + a^4 - 1 = 0.$

22. $4a^2x = (a^2 - b^2 + x)^2.$

23. $1 - 4\sqrt{x} - \sqrt{7x+2} = 0.$ (1)

Transposing $\sqrt{7x+2}$ and squaring, we obtain

$$1 - 8\sqrt{x} + 16x = 7x + 2,$$

or

$$9x - 1 = 8\sqrt{x}. \qquad (2)$$

Squaring (2), $81x^2 - 18x + 1 = 64x.$

$$\therefore\ 81x^2 - 82x + 1 = 0.$$

$$\therefore\ (x-1)(81x-1) = 0. \qquad (3)$$

The roots of (3) are evidently 1 and $\frac{1}{81}$. Hence, if (1) has any root, it must be 1 or $\frac{1}{81}$. But neither of these roots satisfy (1); hence (1) has no root, or is impossible.

It should be noted, however, that if we use both the positive and negative values of $\sqrt{x}$ and $\sqrt{7x+2}$, we obtain in addition to (1) the three equations

$$1 - 4\sqrt{x} + \sqrt{7x+2} = 0, \qquad (4)$$

$$1 + 4\sqrt{x} - \sqrt{7x+2} = 0, \qquad (5)$$

$$1 + 4\sqrt{x} + \sqrt{7x+2} = 0. \qquad (6)$$

Multiplying together the first members of (1), (4), (5), and (6), we obtain the first member of (3); hence equation (3) is equivalent to the four equations (1), (4), (5), (6). Now 1 is a root of (4), and $\frac{1}{81}$ is a root of (5), but neither is a root of (6); hence (6) is impossible. Equation (3) could be obtained from (4), (5), or (6) in the same way it was from (1).

24. $2\sqrt{4 + \sqrt{2x^3 + x^2}} = x + 4.$

25. $x\sqrt{6 + x^2} = 1 + x^2.$

26. $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$.

27. $\dfrac{1}{x+\sqrt{2-x^2}}+\dfrac{1}{x-\sqrt{2-x^2}}=\dfrac{x}{2}$.

28. $x+\sqrt{1+x^2}=\dfrac{2}{\sqrt{1+x^2}}$.

29. $\dfrac{5(3x-1)}{1+5\sqrt{x}}+\dfrac{2}{\sqrt{x}}=3\sqrt{x}$.

30. $\dfrac{x-8}{x-5}+\dfrac{2(x+8)}{x+4}=\dfrac{3x+10}{x+1}$.

31. $\dfrac{12}{5-x}+\dfrac{4}{4-x}=\dfrac{32}{x+2}$.

32. $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}=a$.

33. $mx^2-\dfrac{m^3-n^2}{mn}x=1$.

34. $\dfrac{x^2}{3m-2a}-\dfrac{x}{2}=\dfrac{m^2-4a^2}{4a-6m}$.

35. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

36. $\dfrac{x^2}{\sqrt{a}+\sqrt{b}}-(\sqrt{a}-\sqrt{b})\,x=\dfrac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}$.

37. If the equation $x^2-15-m(2x-8)=0$ has equal roots, find the value of m.

38. Prove that the roots of the following equations are real:

(1) $x^2 - 2ax + a^2 - b^2 - c^2 = 0.$

(2) $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0.$

39. For what values of m will the equation

$$x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$$

have equal roots?

40. Prove that the roots of the equation

$$(a + c - b)x^2 + 2cx + (b + c - a) = 0$$

are rational.

41. For what value of m will the equation

$$\frac{x^2 - bx}{ax - c} = \frac{m - 1}{m + 1}$$

have roots arithmetically equal, but opposite in sign?

Solve the following equations:

42. $x^3 = 1.$

43. $x^3 + 1 = 0.$

44. $x^4 - 1 = 0.$

45. $x^4 + 1 = 0.$

46. $x^6 - 1 = 0.$

47. $x^6 + 1 = 0.$

48. $x^3 + x^2 - 4x - 4 = 0.$

49. $(x^2 - 8x + 2)(x^2 + 2x + 7) = 0.$

Resolve into factors the following trinomials:

50. $4x^2 - 15x + 3$.

51. $5x^2 - 11x + 18$.

52. $7x^2 + 15x + 13$.

53. $3x^2 + 12x + 15$.

153. Higher Equations Solved as Quadratics. Higher equations may frequently be solved as quadratics by a judicious grouping of the terms containing the unknown quantity, so that one group shall be the square of the other.

EXAMPLE. Solve $x^4 - 6x^3 + 5x^2 + 12x = 6$. (1)

Adding and subtracting $4x^2$, we may write (1) in the form

$$(x^4 - 6x^3 + 9x^2) - (4x^2 - 12x) - 60 = 0,$$

or

$$(x^2 - 3x)^2 - 4(x^2 - 3x) - 60 = 0.$$

$$\therefore\ x^2 - 3x = 2 \pm 8. \qquad (2)$$

The solution is now reduced to the solution of the two quadratic equations given in (2).

EXERCISE 11.

Solve the following equations:

1. $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$.

2. $x^4 - 8x^3 + 29x^2 - 52x + 36 = 126$.

3. $x^3 - 6x^2 + 11x = 6$.

Multiply both members by x, thus introducing the root zero; but this must not be included among the roots of the given equation.

4. $x^4 - 2x^3 + x = 380.$

5. $x^4 - 4x^3 + 8x^2 - 8x = 21.$

6. $4x^4 + \frac{1}{2}x = 4x^3 + 33.$

7. $x + 16 - 7\sqrt{x + 16} = 10 - 4\sqrt{x + 16}.$

8. $2x^2 - 2x + 2\sqrt{2x^2 - 7x + 6} = 5x - 6.$

9. $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{1}{2}(3x + 33).$

10. $x^4 + 4x^2 = 12.$

11. $x^4 = 81.$

12. $ax^{\frac{3}{2}} + bx^{\frac{3}{4}} = c.$

13. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}.$

14. $x^{\frac{2}{n}} + 6 = 5x^{\frac{1}{n}}.$

15. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} - 2 = 0.$

16. $1 + 8x^{\frac{6}{5}} + 9\sqrt[5]{x^3} = 0.$

17. $3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x.$

18. $8 + 9\sqrt{(3x - 1)(x - 2)} = 3x^2 - 7x.$

19. $x^2 + 3 - \sqrt{2x^2 - 3x + 2} = \frac{3}{2}(x + 1).$

20. $x^2 + \dfrac{1}{x^2} + 2\left(x + \dfrac{1}{x}\right) = \dfrac{142}{9}.$

21. $\sqrt{x + 12} + \sqrt[4]{x + 12} = 6.$

CHAPTER VIII.

SYSTEMS OF EQUATIONS.

154. A *single* equation involving two or more unknown quantities admits of an infinite number of solutions.

Thus, of $y = 2x + 3$, one solution is $x = 1, y = 5$; another is $x = 2$, $y = 7$; another is $x = 3$, $y = 9$; and so on. In fact, whatever value is given to x, y has a corresponding value.

Of $y = 4x - 1$, one solution is $x = 1, y = 3$; another is $x = 2$, $y = 7$; another is $x = 3$, $y = 11$; and so on.

Both equations have the solution $x = 2$, $y = 7$, which is therefore a solution of the two equations.

155. Equations which are to be satisfied by the same set or sets of values of their unknown quantities are said to be **Simultaneous.**

Simultaneous equations which express different relations between the unknown quantities are said to be **Independent.** Of two or more independent equations, no one can be obtained from one or more of the others.

Thus, of the simultaneous equations (1), (2), and (3), any two are independent, since no one can be obtained from another. But the three are not independent, for any one of them can readily be obtained from the other two.

$$x - 2y + 3z = 2, \qquad (1)$$

$$2x - 3y + z = 1, \qquad (2)$$

$$3x - 5y + 4z = 3. \qquad (3)$$

Thus, by adding (1) and (2), we obtain (3); and by subtracting (2) from (3), we obtain (1).

156. A **System** of equations is a group of two or more independent simultaneous equations.

157. A **Solution** of a system of equations is any set of values of the unknown quantities which will satisfy each of the equations.

158. Two systems of equations are **Equivalent** when they have the same solutions; that is, when every solution of either system is a solution of the other.

159. If each equation of a system contains only one unknown quantity, the system is solved by previous methods. But if each equation contains two or more unknown quantities, we must combine the equations of the system so as to obtain finally an equivalent system in which each equation contains only one unknown quantity. This process is called **Elimination**, and is dependent on the following principles.

160. *The equation obtained by adding or subtracting any two equations of a system may be substituted for either one of them.*

Let $A = A'$ and $B = B'$ be any two equations in x, y, z, ...; then the systems (a) and (b) are equivalent.

$$\left.\begin{array}{r} A = A' \\ B = B' \end{array}\right\} (a) \qquad \left.\begin{array}{c} A = A' \\ A \pm B = A' \pm B' \end{array}\right\} (b)$$

For it is evident that systems (a) and (b) are each satisfied by any set of values of x, y, z, ..., that will render A equal to A' and B to B', and neither is satisfied by any other set. Hence systems (a) and (b) are equivalent.

Either of the equations of the given system may evidently be multiplied by any known quantity before they are added or subtracted.

161. Elimination by Addition or Subtraction. This method is based on the principle of § 160. We will illustrate it by two examples.

EXAMPLE 1. Solve

$$\left.\begin{array}{rr} 3x + 8y = 25, & (1) \\ 12x - 7y = 22. & (2) \end{array}\right\} (a)$$

Multiplying (1) by 7, $\quad 21x + 56y = 175. \quad (3)$

Multiplying (2) by 8, $\quad 96x - 56y = 176. \quad (4)$

Adding (3) and (4), $\quad 117x = 351,$

or

$$\left.\begin{array}{rr} x = 3. & (5) \\ y = 2. & (6) \end{array}\right\} (b)$$

Similarly, we obtain $y = 2$. (6)

By § 160, (5) and (6) may be substituted for (1) and (2), respectively; hence the solution of system (a) is given in (b).

EXAMPLE 2. Solve

$$\left.\begin{aligned} \frac{m}{x}+\frac{n}{y}&=c, && (1)\\ \frac{m'}{x}+\frac{n'}{y}&=c'. && (2) \end{aligned}\right\}(a)$$

Multiplying (1) by n', $\quad \dfrac{mn'}{x}+\dfrac{nn'}{y}=cn'.\quad (3)$

Multiplying (2) by n, $\quad \dfrac{m'n}{x}+\dfrac{nn'}{y}=c'n.\quad (4)$

Subtracting (4) from (3), $\quad (mn'-m'n)\dfrac{1}{x}=cn'-c'n,$

or

$$\left.\begin{aligned} x&=\frac{mn'-m'n}{cn'-c'n}.\\ \text{Similarly, we obtain}\qquad y&=\frac{m'n-mn'}{cm'-c'm}. \end{aligned}\right\}(b)$$

By § 160 the solution of system (a) is given in (b).

162. *If* (a) *be any system of equations in which* A *does not contain* x, *and* (b) *a system obtained from system* (a) *by substituting* A *for* x *in equations* (2) *and* (3), *then the systems* (a) *and* (b) *are equivalent.*

$$\left.\begin{aligned} x&=A. && (1)\\ B&=B'. && (2)\\ C&=C'. && (3) \end{aligned}\right\}(a) \qquad \left.\begin{aligned} x&=A. && (1)\\ B_1&=B_1'. && (2')\\ C_1&=C_1'. && (3') \end{aligned}\right\}(b)$$

Any solution of system (a) will evidently satisfy (2) and (3) after A has been substituted for its equal, x; hence any solution of (a) is a solution of (b). Again, any solution of (b) will evidently satisfy ($2'$) and ($3'$) after x has been substituted for its equal, A; hence any solution of (b) is a solution of (a). Therefore systems (a) and (b) are equivalent.

163. The method of **Elimination by Substitution** is based on the principle of § 162. We will illustrate the method by a single example.

EXAMPLE. Solve

$$\left.\begin{aligned} 3x+2y+4z&=19, &(1)\\ 2x+5y+3z&=21, &(2)\\ 3x-y+z&=4. &(3)\end{aligned}\right\}(a)$$

From (3), $y=3x+z-4.$ (4)

Substituting in (1) and (2) the value of y in (4), we obtain

$$3x+2(3x+z-4)+4z=19,$$

and $$2x+5(3x+z-4)+3z=21;$$

or and

$$\left.\begin{aligned} 9x+6z&=27, &(5)\\ 17x+8z&=41. &(6)\end{aligned}\right\}(b)$$

From (5), and from (6) and (7),

$$\left.\begin{aligned} x&=\frac{27-6z}{9}. &(7)\\ \frac{459-102z}{9}+8z&=41. &(8)\end{aligned}\right\}(c)$$

From (8), and from (7) and (9),

$$\left.\begin{aligned} z&=3. &(9)\\ x&=1. &(10)\end{aligned}\right\}(d)$$

From (4), (9), and (10), $y=2.$ (11)

By § 162 the systems (b), (c), and (d) are equivalent. But (b) with (4) form a system equivalent to (a); hence (d) with (4), or (d) with (11), form a system equivalent to (a). Hence the solution of system (a) is $x=1$, $y=2$, $z=3$.

164. If in the method of elimination by substitution, each of the equations is solved for the same unknown quantity before the substitutions are made, the method is called **Elimination by Comparison.**

165. The following modification of the method of elimination by addition is called **Elimination by Undetermined Multipliers.**

EXAMPLE. Solve

$$\left.\begin{aligned} ax + by &= c, \quad &(1)\\ a'x + b'y &= c'. \quad &(2)\end{aligned}\right\}(a)$$

Multiplying (2) by m, and adding the resulting equation to (1), we obtain,

$$(a + ma')x + (b + mb')y = c + c'm. \qquad (3)$$

To find x, let m be determined by the equation

$$b + mb' = 0;$$

hence

$$m = -b \div b'.$$

Substituting this value of m in (3), we obtain

$$\left(a - \frac{a'b}{b'}\right)x = c - \frac{c'b}{b'}; \qquad (4)$$

hence

$$x = \frac{b'c - bc'}{ab' - a'b}. \qquad (5)$$

By introducing into equation (3) the condition

$$a + ma' = 0, \text{ or } m = -a \div a',$$

we obtain

$$y = \frac{a'c - ac'}{a'b - ab'}. \qquad (6)$$

By § 160, (5) and (6) may be substituted for (1) and (2), respectively; that is, the solution of (a) is given in (5) and (6). Since (1) and (2) are general equations of the first degree between x and y, (5) and (6) may be used as general formulas for solving any system of simple equations in x and y.

Hence, a system of two linear equations in x and y has in general one, and only one, solution.

EXERCISE 12.

Solve the following systems of linear equations:

1. $6x + 4y = 236,$

$3x + 15y = 573.$

2. $ax + by = a^2,$

$bx + ay = b^2.$

3. $\frac{x}{m} + \frac{y}{m'} = 1,$

$\frac{x}{m'} - \frac{y}{m} = 1.$

4. $\frac{5}{x} + \frac{3}{y} = 30,$

$\frac{9}{x} - \frac{5}{y} = 2.$

5. $(a - b)x + (a + b)y = 2(a^2 - b^2),$

$$ax - by = a^2 + b^2.$$

6. $\frac{m}{nx} + \frac{n}{mx} = m + n,$

$\frac{n}{x} + \frac{m}{y} = m^2 + n^2.$

7. $2x + 3y + 4z = 29,$

$3x + 2y + 5z = 32,$

$4x + 3y + 2z = 25.$

$ay + bx = c,$

$cx + az = b,$

$bz + cy = a.$

9. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 36,$

$\frac{1}{x} + \frac{3}{y} - \frac{1}{z} = 28,$

$\frac{1}{x} + \frac{1}{3y} + \frac{1}{2z} = 20.$

10. $3z + 8u = 33,$

$7x - 2z + 3u = 17,$

$4y - 2z + v = 11,$

$4y - 3u + 2v = 9,$

$5y - 3x - 2u = 8.$

SYSTEMS OF QUADRATIC EQUATIONS.

166. *A system consisting of one simple and one quadratic equation has in general two, and only two, solutions.*

This theorem will become evident from the solution of the following example:

EXAMPLE. Solve

$$\left.\begin{aligned} 8x - 4y &= -12, &(1)\\ 3x^2 + 2y^2 - y &= 48. &(2)\end{aligned}\right\}(a)$$

Solving (1) for y, From (2) and (3),

$$\left.\begin{aligned} y &= 2x + 3. &(3)\\ x^2 + 2x &= 3. &(4)\end{aligned}\right\}(b)$$

From (4), From (3) and (5),

$$\left.\begin{aligned} x &= 1, \text{ or } -3. &(5)\\ y &= 5, \text{ or } -3. &(6)\end{aligned}\right\}(c)$$

From § 162 the systems (a), (b), and (c) are equivalent; hence the two solutions of (a) are given in (c).

167. A system of two complete quadratic equations in x and y has in general *four* solutions.

Any such system can evidently be reduced to the form

$$\left.\begin{aligned} x^2 + bxy + cy^2 + dx + ey + f &= 0, &(1)\\ x^2 + b'xy + c'y^2 + d'x + e'y + f' &= 0. &(2)\end{aligned}\right\}(a)$$

Subtracting (2) from (1), and solving the resulting equation for x, we obtain

$$x = \frac{(c - c')y^2 + (e - e')y + f - f'}{(b' - b)y + d' - d}. \qquad (3)$$

Substituting in (1) the value of x in (3), we shall evidently obtain a general equation of the fourth degree in y. This

equation will give four, and only four, values for y (see example of § 153). For each value of y in (3), x has one, and only one, value. Hence a system of two complete quadratics has in general *four* solutions. If, however, $c = c'$ and $b = b'$, (3) will be a linear equation in x and y, and therefore system (a) will be equivalent to one consisting of a linear and a quadratic equation; hence in this case the system has but *two* solutions.

168. By § 167 the solution of a system of two complete quadratics involves in general the solution of a complete biquadratic, of which we have not yet obtained the general solution. But many systems of incomplete quadratics can be solved by the methods of quadratics. We shall next consider some of the most useful methods of solving systems involving incomplete equations of the second and higher degrees.

169. When the system is of the form

$$\left.\begin{aligned} x \pm y &= a \\ xy &= b \end{aligned}\right\}, \qquad \left.\begin{aligned} x^2 + y^2 &= a \\ xy &= b \end{aligned}\right\}, \text{ or } \left.\begin{aligned} x^2 + y^2 &= a \\ x + y &= b \end{aligned}\right\},$$

it may be solved symmetrically by finding the values of $x + y$ and $x - y$.

EXAMPLE 1. Solve

$$\left.\begin{aligned} x - y &= 4, \quad (1) \\ xy &= 60. \quad (2) \end{aligned}\right\}(a)$$

From (1), $$x^2 - 2xy + y^2 = 16. \quad (3)$$

Multiplying (2) by 4, $$4xy = 240. \quad (4)$$

Adding (3) and (4), $$x^2 + 2xy + y^2 = 256,$$

or $$x + y = \pm 16. \quad (5)$$

Now (1) and (5) are equivalent to the two systems

$$\left.\begin{aligned} x-y &= 4, \\ x+y &= 16; \end{aligned}\right\} \qquad \left.\begin{aligned} x-y &= 4, \\ x+y &= -16. \end{aligned}\right\} \quad (b)$$

Whence

$$\left.\begin{aligned} x &= 10, \\ y &= 6; \end{aligned}\right\} \qquad \left.\begin{aligned} x &= -6, \\ y &= -10. \end{aligned}\right\} \quad (c)$$

The values in (c) evidently satisfy system (a). Equations (2) and (5) would form a system having other solutions than those of system (a), introduced by squaring (1).

EXAMPLE 2. Solve

$$\left.\begin{aligned} x^2+y^2 &= 65, \quad (1) \\ xy &= 28. \quad (2) \end{aligned}\right\} \quad (a)$$

Multiplying (2) by 2, then adding and subtracting, we have

$$x^2+2xy+y^2 = 121,$$

and

$$x^2-2xy+y^2 = 9;$$

or

$$\left.\begin{aligned} x+y &= \pm 11, \\ x-y &= \pm 3. \end{aligned}\right\} \quad (b)$$

and

Now (b) is equivalent to the four systems

$$\left.\begin{aligned} x+y &= 11, \\ x-y &= 3; \end{aligned}\right\} \left.\begin{aligned} x+y &= 11, \\ x-y &= -3; \end{aligned}\right\} \left.\begin{aligned} x+y &= -11, \\ x-y &= 3; \end{aligned}\right\} \left.\begin{aligned} x+y &= -11, \\ x-y &= -3. \end{aligned}\right\} \quad (c)$$

Whence

$$\left.\begin{aligned} x &= 7, \\ y &= 4; \end{aligned}\right\} \quad \left.\begin{aligned} x &= 4, \\ y &= 7; \end{aligned}\right\} \quad \left.\begin{aligned} x &= -4, \\ y &= -7; \end{aligned}\right\} \quad \left.\begin{aligned} x &= -7, \\ y &= -4. \end{aligned}\right\} \quad (d)$$

By § 160 systems (a) and (b) are equivalent, so also are systems (c) and (d).

Any system of equations of the form

$$\left.\begin{aligned} x^2 \pm pxy + y^2 &= a^2, \\ x \pm y &= b, \end{aligned}\right\}$$

in which p is any numerical quantity, can evidently be reduced to the first form given above.

170. Terms of Even Degree. When all the terms of two simultaneous quadratic equations are of an even degree, the system can be solved as follows:

EXAMPLE. Solve
$$\left.\begin{aligned} x^2+xy+2y^2&=44, &(1)\\ 2x^2-xy+y^2&=16. &(2)\end{aligned}\right\}(a)$$

First Solution. Let
$$\left.\begin{aligned} y&=vx. &(3)\\ x^2(1+v+2v^2)&=44. &(4)\\ x^2(2-v+v^2)&=16. &(5)\end{aligned}\right\}(b)$$
(4): From (1) and (3); (5): From (2) and (3).

Equating the two values of x^2 obtained from (4) and (5), and clearing of fractions, we obtain

$$v^2-5v=-6.$$

Hence $v=3$, or 2.

From (5) and $v=3$, we obtain

$$\left.\begin{aligned} x&=+\sqrt{2},-\sqrt{2}. &(6)\\ y&=+3\sqrt{2},-3\sqrt{2}. &(7)\end{aligned}\right\}(c)$$
(7): From (3) and (6).

From (5) and $v=2$, we obtain

$$\left.\begin{aligned} x&=+2,-2. &(8)\\ y&=+4,-4. &(9)\end{aligned}\right\}(d)$$
(9): From (3) and (8).

Systems (*a*) and (*b*) are equivalent; hence the four solutions of system (*a*) are given in (*c*) and (*d*).

Second Solution.

Multiplying (1) by 4, $4x^2+4xy+8y^2=176.$ (3)

Multiplying (2) by 11, $22x^2-11xy+11y^2=176.$ (4)

Subtracting (3) from (4), $18x^2-15xy+3y^2=0,$

or
$$\left(\frac{y}{x}\right)^2-5\left(\frac{y}{x}\right)+6=0.$$

Hence $\dfrac{y}{x}=3$, or 2. (5)

Now (1) and (5) are equivalent to the two systems

$$\left.\begin{aligned} x^2+xy+2y^2&=44,\\ y&=3x;\end{aligned}\right\} \qquad \left.\begin{aligned} x^2+xy+2y^2&=44,\\ y&=2x;\end{aligned}\right\}$$

which are readily solved.

EXERCISE 13.

Solve the following systems of equations:

1. $x + y = 51,$
$xy = 518.$

2. $x - y = 18,$
$xy = 1075.$

3. $x - y = 4,$
$x^2 + y^2 = 106.$

4. $x^2 + y^2 = 178,$
$x + y = 16.$

5. $x - y = 3,$
$x^2 - 3xy + y^2 = -19.$

6. $x^2 - xy + y^2 = 76,$
$x + y = 14.$

7. $x^2 + xy + y^2 = 67,$
$x^2 - xy + y^2 = 39.$

8. $x^2 - 2xy - y^2 = 1,$
$x + y = 2.$

9. $3x^2 - 2y^2 + 5x - 2y = 28,$
$x + y + 4 = 0.$

10. $x^2 - 3xy + y^2 = -1,$
$3x^2 - xy + 3y^2 = 13.$

11. $3x^2 - 5y^2 = 28,$
$3xy - 4y^2 = 8.$

12. $x^2 + 3xy = 54,$
$xy + 4y^2 = 115.$

13. $x^2y^2 + 5xy = 84,$
$x + y = 8.$

14. $x^2 + y^2 - 3 = 3xy,$
$2x^2 - 6 + y^2 = 0.$

15. $3x^2 + xy + y^2 = 15,$
$31xy - 3x^2 - 5y^2 = 45.$

16. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$
$x + y = 6.$

17. $\frac{1}{x} - \frac{1}{y} = \frac{1}{3},$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{9}.$

18. $x^2 - 2xy - y^2 = 31,$
$\frac{1}{2}x^2 + 2xy - y^2 = 101.$

19. $x^2y^2 + 4xy = 12,$
$x^2 - xy + 4y^2 = 8.$

171. When x and y are *symmetrically* involved in two simultaneous equations, the system can frequently be solved by putting $x = v + w$ and $y = v - w$.

EXAMPLE. Solve

$$\left.\begin{aligned} x^4 + y^4 &= 82, \quad &(1)\\ x - y &= 2. \quad &(2)\end{aligned}\right\}(a)$$

$$\left.\begin{aligned} &\text{Put} & x &= v + w, &(3)\\ &\text{and} & y &= v - w. &(4)\\ &\text{From (2), (3), and (4),} & w &= 1. &(5)\\ &\text{From (1), (3), (4), and (5),} & &(v+1)^4 + (v-1)^4 = 82, &\\ &\text{or} & v &= \pm 2, \text{ or } \pm\sqrt{-10}. &(6)\end{aligned}\right\}(b)$$

$$\left.\begin{aligned} &\text{From (3), (5), and (6),} & x &= 3, -1, 1 \pm \sqrt{-10}.\\ &\text{From (4), (5), and (6),} & y &= 1, -3, -1 \pm \sqrt{-10}.\end{aligned}\right\}(c)$$

172. *In general, system* (a) *is equivalent to the two systems* (b) *and* (c).

$$\left.\begin{aligned} AB &= A'B',\\ B &= \quad B'.\end{aligned}\right\}(a) \qquad \left.\begin{aligned} A &= A',\\ B &= B'.\end{aligned}\right\}(b) \qquad \left.\begin{aligned} B &= 0,\\ B' &= 0.\end{aligned}\right\}(c)$$

Since $B = B'$, $AB = A'B'$ is equivalent to

$$B(A - A') = 0,$$

or

$$A = A', B = 0.$$

Hence system (a) is equivalent to (b) and (c).

The first equation in system (b) is obtained by dividing the first equation in (a) by the second.

If B and B' cannot each be zero, system (c) is impossible, and system (b) is equivalent to system (a).

This principle is often useful in solving systems involving higher equations.

EXAMPLE. Solve
$$\left.\begin{aligned} x^4+x^2y^2+y^4 &= 7371, & (1)\\ x^2-xy+y^2 &= 63. & (2)\end{aligned}\right\}(a)$$

Dividing (1) by (2), $x^2+xy+y^2 = 117.$ (3)

$$\left.\begin{aligned} &\text{Adding (2) to (3),} & x^2+y^2 &= 90. & (4)\\ &\text{Subtract (2) from (3),} & 2xy &= 54. & (5)\end{aligned}\right\}(b)$$

Hence
$$\left.\begin{aligned} x &= +9, -9, +3, -3,\\ y &= +3, -3, +9, -9.\end{aligned}\right\}(c)$$

Here $B' = 63$, and the system, $B = 0$, $B' = 0$, is impossible; hence equations (2) and (3) form a system equivalent to (a). Therefore all the solutions of (a) are given in (c).

EXERCISE 14.

Solve the following systems of equations:

1. $x^3 + y^3 = 637,$
   $x + y = 13.$

2. $x^3 + y^3 = 126,$
   $x^2 - xy + y^2 = 21.$

3. $x^4 + x^2y^2 + y^4 = 2128,$
   $x^2 + xy + y^2 = 76.$

4. $x + y - \sqrt{xy} = 7,$
   $x^2 + y^2 + xy = 133.$

5. $x + y = 4,$
   $x^4 + y^4 = 82.$

6. $\dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{9}{2},$
   $\dfrac{3}{x+y} = 1.$

7. $x^3 - y^3 = 56,$
   $x^2 + xy + y^2 = 28.$

8. $xy(x + y) = 30,$
   $x^3 + y^3 = 35.$

9. $x^3 - y^3 = 127,$
   $x^2y - xy^2 = 42.$

10. $5x^2 - 5y^2 = x + y,$
    $3x^2 - 3y^2 = x - y.$

11. $x^4 + y^4 = 272,$
    $x - y = 2.$

12. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2},$
    $x^2 + y^2 = 20.$

CHAPTER IX.

INDETERMINATE EQUATIONS AND SYSTEMS, DISCUSSION OF PROBLEMS, INEQUALITIES.

173. An **Impossible Equation** or **System of Equations** is one that has no finite solution. Such an equation or system involves some absurdity.

Two equations are said to be **Inconsistent** when they express relations between the unknown quantities that cannot coexist. Any system that contains inconsistent equations or embraces more independent equations than unknown quantities is impossible.

Thus, $\frac{5}{6}x + \frac{1}{4}x - 5 = \frac{1}{2}x + \frac{7}{12}x + 8$ is an impossible equation; for attempting to solve it, we obtain the absurdity $0 = 312$.

The system
$$\begin{cases} ax + by = c, & (1) \\ 3ax + 3by = 5c, & (2) \end{cases}$$
is impossible; for attempting to solve it, we obtain the absurdity $3 = 5$.

Equations (1) and (2) are evidently inconsistent.

The system
$$\begin{cases} x + y = 9, & (1) \\ 2x + y = 13, & (2) \\ x + 2y = 16, & (3) \end{cases}$$
is impossible, for solving (1) and (2), we obtain $x = 4$, $y = 5$; substituting these values in (3), we obtain not an identity, but the absurdity, $14 = 16$. Similarly the solution of any other two of these three equations will not satisfy the third.

Again, if in the system

$$\left.\begin{aligned} a^2x^2 - b^2y^2 &= c^2, \quad &(1)\\ ax - by &= c, \quad &(2)\end{aligned}\right\}(a)$$

we divide equation (1) by (2), we obtain

$$ax + by = c. \quad (3)$$

Now (2) and (3) form a system equivalent to system (a); hence (a) has but one solution. But by § 166 such a system as (a) has in general two solutions. System (a) is called a *defective* system.* Nearly all the systems in Exercise 14 are defective.

174. An **Indeterminate Equation** or **System of Equations** is one that admits of an infinite number of solutions. Hence a single equation containing two or more unknown quantities is indeterminate (§ 154). Again, any system of equations that contains more *unknown* quantities than *independent* equations is indeterminate.

Thus, the system

$$\left.\begin{aligned} ax + by + cz &= 0,\\ a'x + b'y + c'z &= 0,\end{aligned}\right\}$$

is indeterminate; for solving the system for y and z we may give to x any value, and find the corresponding values of y and z.

---

* An impossible equation or system of equations is, in general, but the limiting case of a more general equation or system, the solutions of which in the limit become infinity.

Thus, the equation $ax = b$ becomes impossible only when $a = 0$, and then its root $b \div a$ becomes $b \div 0$, or infinity.

It will be seen in § 176 that a system of linear equations becomes impossible only for a certain relation between the coefficients of its equations, which renders both x and y infinite.

Again, the general system

$$\left.\begin{aligned} a^2x^2 - b^2y^2 &= c^2,\\ ax - (b+e)y &= c,\end{aligned}\right\}$$

becomes the defective system (a), only when $e = 0$.

The system $\left\{\begin{array}{l}3x-4y=\ \ 9\\6x-8y=18\end{array}\right\}$ is indeterminate, for its equations are not independent but equivalent.

Again, the system

$$\begin{aligned}2x+3y-\ \ z&=15, \qquad (1)\\3x-\ \ y+2z&=\ \ 8, \qquad (2)\\5x+2y+\ \ z&=23, \qquad (3)\end{aligned}$$

is indeterminate. No two of its equations are equivalent, but any one of them can be obtained from the other two; thus, by adding (1) and (2) we obtain (3). Hence the system contains but two independent equations.

175. Sometimes it is required to find the *positive integral* solutions of an indeterminate equation or system. The following examples will illustrate the simplest general method of finding such solutions.

(1) Solve $7x+12y=220$ in positive integers.

Dividing by 7, the smaller coefficient, expressing improper fractions as mixed numbers, and combining, we obtain

$$x+y+\frac{5y-3}{7}=31. \qquad (1)$$

Since x and y are integers, $31-x-y$ is an integer; hence the fraction in (1), or any integral multiple of it, equals an integer. Multiplying this fraction by such a number as to make the coefficient of y divisible by the denominator with remainder 1, which in this case is 3, we have

$$\frac{15y-9}{7}=2y-1+\frac{y-2}{7}=\text{an integer}.$$

Hence $\quad \dfrac{y-2}{7}=$ an integer $=p$, suppose.

$$\therefore\ y=7p+2. \qquad (2)$$

From (1) and (2), $\qquad x=28-12p. \qquad (3)$

Since x and y are *positive* integers, from (2) it follows that $p > -1$, and from (3), that $p < 3$; hence

$$p = 0, 1, 2. \tag{4}$$

From (2), (3), and (4), we obtain the three solutions

$$x = 28, 16, 4;$$
$$y = 2, 9, 16.$$

(2) Solve in positive integers the system

$$x + y + z = 43, \tag{1}$$
$$10x + 5y + 2z = 229. \tag{2}$$

Eliminating z, $\qquad 8x + 3y = 143,$

or
$$y + 2x + \frac{2x-2}{3} = 47. \tag{3}$$

$$\therefore \frac{4x-4}{3} = x - 1 + \frac{x-1}{3} = \text{an integer}.$$

$$\therefore \frac{x-1}{3} = \text{an integer} = p, \text{ suppose}.$$

$$\therefore x = 3p + 1. \tag{4}$$

From (3) and (4), $\qquad y = 45 - 8p. \qquad (5)$

From (1), (4), and (5), $\quad z = 5p - 3. \qquad (6)$

From (6), $p > 0$; and from (5), $p < 6$; hence

$$p = 1, 2, 3, 4, 5.$$

Whence
$$x = 4, 7, 10, 13, 16;$$
$$y = 37, 29, 21, 13, 5;$$
$$z = 2, 7, 12, 17, 22.$$

Thus, the system has five positive integral solutions.

(3) Show that $ax + by = c$ has no integral solutions, if a and b have a common factor not a divisor of c.

Let $a = md$, $b = nd$, c not containing d; then

$$mdx + ndy = c,$$

or

$$mx + ny = c \div d. \qquad (1)$$

Now $c \div d$ is an irreducible fraction; while $mx + ny$ is an integer for any integral solution of (1); hence (1) cannot have an integral solution.

EXERCISE 15.

Solve in positive integers

1. $3x + 29y = 151.$
2. $3x + 8y = 103.$
3. $7x + 12y = 152.$
4. $13x + 7y = 408.$
5. $23x + 25y = 915.$
6. $13x + 11y = 414.$
7. $6x + 7y + 4z = 122,$ $\quad 11x + 8y - 6z = 145.$
8. $12x - 11y + 4z = 22,$ $\quad -4x + 5y + z = 17.$
9. $20x - 21y = 38,$ $\quad 3y + 4z = 34.$
10. $5x - 14y = 11.$
11. $13x + 11z = 103,$ $\quad 7z - 5y = 4.$
12. $14x - 11y = 29.$

13. A farmer buys horses at \$111 a head, cows at \$69, and spends \$2256; how many of each does he buy?

14. A drover buys sheep at \$4, pigs at \$2, and oxen at \$17; if 40 animals cost him \$301, how many of each kind does he buy?

15. I have 27 coins, which are dollars, half-dollars, and dimes, and they amount to \$9.80; how many of each sort have I?

176. The Symbols $\frac{0}{a}$, $\frac{a}{0}$, and $\frac{0}{0}$. The symbol 0 denotes absolute zero; that is, a denoting any number, $0 = a - a$.

As a quotient, the symbol $0 \div a$ denotes that number which multiplied by a equals zero; hence $0 \div a = 0$.

As a quotient, the symbol $a \div 0$ denotes that number which multiplied by zero equals a; but any number, however large, multiplied by zero cannot exceed zero. For this reason the symbol $\frac{a}{0}$ represents that which transcends all quantity, or ***absolute infinity.***

As a quotient, the symbol $0 \div 0$ denotes the number which multiplied by zero equals zero; but any number whatever multiplied by zero equals zero. Hence the symbol $\frac{0}{0}$ represents any number whatever. For this reason $\frac{0}{0}$ is called the **Symbol of Indetermination.**

It will be seen, however, in § 235 that an expression may assume this indeterminate form and still have a determinate value.

EXAMPLE. By discussing its solution show that the system

$$\left.\begin{aligned} ax + by &= c, \\ a'x + b'y &= c', \end{aligned}\right\} \qquad (a)$$

is (i.) indeterminate if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$; $\qquad$ (1)

and (ii.) impossible if $\frac{a}{a'} = \frac{b}{b'}$ not $= \frac{c}{c'}$. $\qquad$ (2)

By Example of § 165 the solution of system (a) is

$$x = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{ac' - a'c}{ab' - a'b}. \tag{3}$$

(i.) If relation (1) exists, then from (1) we have $ab' - a'b = 0$, $b'c - bc' = 0$, $ac' - a'c = 0$; hence the values of x and y in (3) each assume the form $\frac{0}{0}$, and the system has an infinite number of solutions.

(ii.) If relation (2) exists, then $ab' - a'b = 0$, but neither $b'c - bc'$ nor $ac' - a'c$ is zero; hence x and y are infinite, and the system is impossible.

It is evident that the equations in (a) are equivalent when (1) is satisfied, and inconsistent when (2) holds true.

177. Solution of Problems. The *Algebraic Solution* of a problem consists of three distinct parts: (1) The *Statement* in equations; (2) the *Solution* of these equations; and (3) the *Discussion* of this solution.

To *State* a problem in equations is to express by one or more equations the relations which exist between its known and unknown quantities; that is, to *translate* the given problem into algebraic language.

Discussion. The problem given may impose on the unknown quantities certain conditions that cannot be expressed by equations. In such cases the solution of the equation or system of equations may not be the solution of the problem.

When the known quantities are represented by letters it may happen that the solution of the equation

or system is the solution of the problem only when the values of the known quantities lie between certain limits.

Again, for certain values of the known quantities, the problem may be indeterminate, that is, have an infinite number of solutions; for certain other values the problem may be impossible.

To discover these and other similar facts when they exist, and to interpret negative results when they occur, is to *discuss* the solution.

Negative solutions denote, in general, the opposite of positive ones. If the problem does not admit of quantities opposite in quality, negative solutions of the equation or system are not solutions of the problem.

(1) In a company of 10 persons a collection is taken; each man gives \$6, each woman \$4. The amount received is \$45. Find the number of men and the number of women.

Let x and y denote the number of men and women, respectively;

then and
$$\left.\begin{aligned} x + y &= 10, \\ 6x + 4y &= 45. \end{aligned}\right\} \qquad (1)$$

$$\therefore \quad x = 2\tfrac{1}{2}, \quad y = 7\tfrac{1}{2}. \qquad (2)$$

Discussion. System (1) translates the problem, and (2) is its only solution; hence the problem can have no other solution than that in (2). But the nature of the problem requires that its solution shall be whole numbers; hence the problem is impossible.

(2) A and B travel in the direction PR at the rates of a and b miles per hour. At 12 o'clock A is at P and B at Q, which is c miles to the right of P. Find when they are together.

$P \qquad\qquad Q \qquad\qquad\qquad\qquad R$

Let distance measured to the right from P, and time reckoned after 12 o'clock, be regarded as positive.

Let $x =$ the number of hours from 12 o'clock to the time of meeting.

Then $$ax = bx + c. \tag{1}$$

Hence $$x = \frac{c}{a-b}. \tag{2}$$

Discussion. If c is not zero, and $a > b$, x is positive; that is, A and B are together at some time after 12 o'clock.

If c is not zero, and $a < b$, x is negative; that is, A and B are together at some time *before* 12 o'clock.

If the problem were to find at what time *after* 12 o'clock A and B are together, this negative solution of (1) would not be a solution of the problem, and the problem would be impossible.

If $c = 0$, and $a > b$ or $a < b$, $x = 0$; that is, A and B are together at 12 o'clock, but not before or after that time.

If c is not zero, and $a = b$, $x = c \div 0$, or absolute infinity; that is, A and B are not together at, before, or after 12 o'clock, and the problem is impossible.

If $c = 0$, and $a = b$, $x = \frac{0}{0}$; that is, there is an infinite number of times when A and B are together, and the problem is indeterminate.

The fraction $\frac{c}{a-b}$ assumes the form $\frac{0}{0}$ by reason of *two independent suppositions;* namely, $c = 0$ and $a = b$. In all such cases a fraction is strictly indeterminate.

INEQUALITIES.

178. The algebraic statement that one quantity is greater or less than another is called an **Inequality**. The signs used are $>$ and its reverse $<$; the opening being toward the greater quantity.

If $a - b$ is *positive*, $a > b$; if $a - b$ is *negative*, $a < b$. The expression $a > b > c$ indicates that b is less than a but greater than c. The expression $a \overline{\overline{>}} b$ indicates that a is either equal to or greater than b.

179. The following principles, used in transforming inequalities, will upon a little reflection become evident:

(i.) An inequality will still hold after both members have been

Increased or diminished by the same quantity;
Multiplied or divided by the same positive quantity;
Raised to any odd power, or to *any* power if both members are essentially positive.

(ii.) The sign of an inequality must be reversed after both members have been

Multiplied or divided by the same negative quantity;
Raised to the same even power, if both members are negative.

(iii.) If the same root be extracted of both members of an inequality, the sign must be reversed only when negative even roots are compared.

180. In establishing the relation of inequality between two symmetrical expressions, the following principle is very useful.

If a *and* b *are unequal and real,* $a^2 + b^2 > 2ab$.

For $$(a-b)^2 > 0,$$

or $$a^2 - 2ab + b^2 > 0; \qquad (1)$$

hence $$a^2 + b^2 > 2ab. \qquad (2)$$

(1) Prove that the arithmetical mean between two unequal quantities is greater than the geometrical mean.

If in (2) we put $a^2 = x$ and $b^2 = y$, we obtain

$$x + y > 2\sqrt{xy}, \text{ or } \frac{x+y}{2} > \sqrt{xy}.$$

(2) Show that $a^3 + b^3 > a^2b + ab^2$, if $a + b > 0$.

From (1), $a^2 - ab + b^2 > ab$.

Multiplying by $a + b$, $a^3 + b^3 > a^2b + ab^2$.

(3) Show that the fraction $\dfrac{a_1 + a_2 + a_3 + \cdots + a_n}{b_1 + b_2 + b_3 + \cdots + b_n} <$ the greatest and $>$ the least of the fractions $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \cdots, \dfrac{a_n}{b_n}$, all the denominators being of the same quality.

Suppose that $\dfrac{a_1}{b_1}$ is the least and $\dfrac{a_n}{b_n}$ the greatest of these fractions, and that the denominators are all positive.

Then $\quad \dfrac{a_1}{b_1} = \dfrac{a_1}{b_1}, \qquad \therefore \quad a_1 = b_1 \times \dfrac{a_1}{b_1};$

$\dfrac{a_2}{b_2} > \dfrac{a_1}{b_1}, \qquad \therefore \quad a_2 > b_2 \times \dfrac{a_1}{b_1};$

$\cdots \quad \cdots \qquad\qquad \cdots \quad \cdots$

$\dfrac{a_n}{b_n} > \dfrac{a_1}{b_1}, \qquad \therefore \quad a_n > b_n \times \dfrac{a_1}{b_1}.$

Adding, $\quad a_1 + a_2 + \cdots + a_n > (b_1 + b_2 + \cdots + b_n)\dfrac{a_1}{b_1}.$

$$\therefore \quad \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} > \frac{a_1}{b_1}.$$

Similarly we may prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} < \frac{a_n}{b_n}.$$

In like manner the principle may be proved when all the denominators are negative.

EXERCISE 16.

1. Show that the sum of any fraction and its reciprocal is greater than 2.

The letters denoting unequal positive numbers, show that

2. $a^2 + b^2 + c^2 > ab + ac + bc;\quad m^3 + 1 > m^2 + m.$

3. $a^3 + b^3 + c^3 > \frac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2).$

4. $\dfrac{a+b}{2} > \dfrac{2ab}{a+b};\quad \dfrac{a}{b^2} + \dfrac{b}{a^2} > \dfrac{1}{b} + \dfrac{1}{a}.$

5. If $a^2 + b^2 + c^2 = 1$, and $m^2 + n^2 + r^2 = 1$, show that $am + bn + cr < 1$.

6. If $5x - 6 < 3x + 8$, and $2x + 1 < 3x - 3$, show that the values of x lie between 4 and 7.

7. If $3x - 2 > \frac{1}{2}x - \frac{4}{5}$, and $\frac{7}{6} - \frac{5}{4}x < 8 - 2x$, show that the values of x lie between $\frac{12}{25}$ and $\frac{82}{9}$.

CHAPTER X.

RATIO, PROPORTION, VARIATION.

181. The **Ratio** of one abstract quantity to another is the quotient of the first divided by the second.

When the quotient $a \div b$ or $\frac{a}{b}$ is spoken of as a ratio, it is often written $a : b$, and read "a *is to* b;" a is called the **Antecedent** and b the **Consequent** of the ratio.

182. By § 48 the value of a ratio is not changed by multiplying or dividing both its antecedent and consequent by the same quantity.

Two ratios may be compared by reducing them as fractions to a common denominator.

183. When two or more ratios are multiplied together they are said to be *compounded*.

Thus the ratio compounded of the three ratios $2 : 3$, $a : d$, and $b : e^2$ is $2ab : 3de^2$.

The ratio $a^2 : b^2$, compounded of the two identical ratios $a : b$ and $a : b$, is called the **Duplicate ratio** of $a : b$. Similarly $a^3 : b^3$ is called the **Triplicate ratio** of $a : b$. Also $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **Subduplicate ratio** of $a : b$.

184. The ratio of two positive quantities is called a ratio of *greater* or *less* inequality according as the antecedent is *greater* or *less* than the consequent.

185. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive quantity to both its terms.*

Let a, b, and x be any positive quantities;

then $$a + x : b + x < \text{or} > a : b,$$

according as $$a > \text{or} < b.$$

For $$\frac{a}{b} - \frac{a+x}{b+x} \equiv \frac{x(a-b)}{b(b+x)}. \qquad (1)$$

Now the second member of (1) is evidently positive or negative according as $a >$ or $< b$.

Hence $\frac{a+x}{b+x} <$ or $> \frac{a}{b}$, according as $a >$ or $< b$.

In like manner it may be proved that *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same quantity from both its terms.*

186. By § 68, $\frac{a}{b} : \frac{c}{d} \equiv \frac{ad}{bc}$.

Hence, unless surds are involved, the ratio of two fractions can be expressed as a ratio of two integers.

If the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be **Commensurable**; otherwise they are said to be **Incommensurable.**

187. Ratio of Concrete Quantities. If A and B be two concrete quantities of the same kind, whose numerical measures in terms of the same unit are a and b, then the ratio of A to B is the ratio of a to b.

If A and B are *incommensurable*, that is, cannot be exactly expressed in terms of the same unit, we can always find two integers whose ratio differs from that of A to B by as little as we please.

For divide B into n equal parts; let β be one of these parts, so that $B = n\beta$. Suppose β is contained in A more than m times and less than $m + 1$ times; then it is axiomatic that

$$A : B > m\beta : n\beta \text{ and } < (m + 1)\beta : n\beta;$$

that is, the ratio $A : B$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

Hence the ratio of A to B differs from that of m to n by less than $1 \div n$, which by increasing n can be made as small as we please.

The ratio of A to B is the *fixed value* to which the ratio of m to n approaches indefinitely near when n is increased without limit.

PROPORTION.

188. Four quantities, a, b, c, d, are said to be *proportional* if the ratio $a : b$ is equal to the ratio $c : d$.

The proportion is written

$$a : b = c : d, \quad a : b :: c : d, \quad \text{or } \frac{a}{b} = \frac{c}{d},$$

and is usually read "a *is to* b *as* c *is to* d."

The first term and the last are called the *Extremes*, and the other two the *Means* of the proportion.

189. Continued Proportion. The quantities a, b, c, d, $\cdots$, are said to be *in continued proportion* if

$$a : b = b : c = c : d = \cdots \qquad (1)$$

In (1), b is said to be a *mean proportional* between a and c, and c a *third proportional* to a and b. Also b and c are said to be *two mean proportionals* between a and d, and so on.

190. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

If $\frac{a}{b} = \frac{c}{d}$, then by § 23 $a\,d = c\,b$.

191. COROLLARY. If $a : b = b : c$, then $b^2 = a\,c$, or $b = \sqrt{a\,c}$.

192. Conversely, *if the product of two quantities equals the product of two other quantities, two of them may be made the extremes and the other two the means of a proportion.*

If $a\,d = c\,b$, then by dividing both members by $b\,d$ we obtain $a : b = c : d$.

193. *If four quantities are in proportion, they are in proportion by*

(i.) *Inversion: If* $a : b = c : d$, *then* $b : a = d : c$.

(ii.) *Alternation: If* $a : b = c : d$, *then* $a : c = b : d$.

(iii.) *Composition: If* $a : b = c : d$, *then* $a + b : b = c + d : d$.

(iv.) *Division: If* $a : b = c : d$, *then* $a - b : b = c - d : d$.

(v.) *Composition and Division: If* $a : b = c : d$, *then* $a + b : a - b = c + d : c - d$.

These propositions and those of §§ 194 and 195 are easily proved by the properties of fractions or by §§ 190 and 192.

194. *If* $a : b = c : d$, *and* $e : f = g : h$, *then* $ae : bf = cg : dh$.

195. *If* $a : b = c : d$, *then*

(i.) $ma : mb = nc : nd$;

(ii.) $ma : nb = mc : nd$;

(iii.) $a^n : b^n = c^n : d^n$, n *being any exponent.*

196. *If we have a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

Let $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots r;$$

then $$a = br,\ c = dr,\ e = fr, \cdots$$

Adding these equations, we obtain

$$a+c+e+\cdots=(b+d+f+\cdots)\,r.$$

$$\therefore \qquad \frac{a+c+e+\cdots}{b+d+f+\cdots}=r=\frac{a}{b}=\cdots$$

REMARK. The method of proof used above might be employed in §§ 193, 194, 195. The proof in the next article further exhibits the directness and simplicity of this method.

197. If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{ma+nb}{pa+qb}=\dfrac{mc+nd}{pc+qd}$.

Let $\dfrac{a}{b}=\dfrac{c}{d}=r$; then $\dfrac{ma+nb}{pa+qb}=\dfrac{mbr+nb}{bpr+qb}=\dfrac{mr+n}{pr+q}$,

and $$\frac{mc+nd}{pc+qd}=\frac{mdr+nd}{pdr+qd}=\frac{mr+n}{pr+q}.$$

$$\therefore \quad \frac{ma+nb}{pa+qb}=\frac{mc+nd}{pc+qd}.$$

198. Proportion of Concrete Quantities. If A, B, be two concrete quantities of the same kind, whose ratio is $a:b$, and C, D, be two other concrete quantities of the same kind (but not necessarily of the same kind as A and B), whose ratio is $c:d$; then

$$A:B=C:D,$$

when $$a:b=c:d.$$

The last proportion can be transformed according to the theory of proportion given above, and the result interpreted with respect to A, B, C, D.

VARIATION.

199. If the relation between y and z is expressed by a single equation, then y and z have an infinite number of sets of values (§ 154), and are called *variables.*

There are an infinite number of ways in which one variable, y, may depend upon another, z. For example, we may have $y = az$, $y = az + c$, $y = az^2 + bz + c$, and so on. In this chapter we shall consider only the simplest relation, $y = az$, in which z denotes any variable, and a is a *constant;* that is, a has the same value for all values of y and z.

200. If $y = az$, the ratio of y to z is the constant a. The expression $y \propto z$ denotes that the ratio of y to z is some constant, and is read "*y varies as z.*" The symbol $\propto$ is called the *Sign of Variation.*

Hence, *if $y = az$, $y \propto z$, and conversely.*

If $y \propto z$, or $y = az$, then any set of values of y and z are proportional to any other set; for the ratio of each set is the constant a.

Hence $y \propto z$ is often read "*y is proportional to z.*"

201. If in § 200, $z = \frac{1}{x}, xv, \frac{x}{v}, x + v$, we have

(i.) *If* $y = a\frac{1}{x}$, *then* $y \propto \frac{1}{x}$, *and conversely.*

$y \propto 1 \div x$ is often read "y varies *inversely* as x."

(ii.) *If* $y = a\,x\,v$, *then* $y \propto x\,v$, *and conversely.*

$y \propto x\,v$ is often read "y varies as x and v *jointly.*"

(iii.) *If* $y = a\frac{x}{v}$, *then* $y \propto \frac{x}{v}$, *and conversely.*

$y \propto x \div v$ is often read "y varies *directly* as x and *inversely* as v."

(iv.) *If* $y = a(x + v)$, *then* $y \propto x + v$, *and conversely.*

202. The simplest method of treating *variations* is to convert them into *equations*. Of the six following propositions we give the proof of the first, and leave the proof of the others as an exercise for the student.

(i.) *If* $u \propto y$, *and* $y \propto x$, *then* $u \propto x$.

By § 200, $u = a\,y$, $y = b\,x$ (a and b being constants), $\therefore u = a\,b\,x$, or $u \propto x$.

(ii.) *If* $u \propto x$, *and* $y \propto x$, *then* $u \pm y \propto x$, *and* $u\,y \propto x^2$.

(iii.) *If* $u \propto x$, *and* $z \propto y$, *then* $u\,z \propto x\,y$.

(iv.) *If* $u \propto x\,y$, *then* $x \propto u \div y$, *and* $y \propto u \div x$.

(v.) *If* $u \propto x$, *then* $z\,u \propto z\,x$.

(vi.) *If* $u \propto x$, *then* $u^n \propto x^n$.

203. *If* $u \propto x$ *when* y *is constant, and* $u \propto y$ *when* x *is constant, then* $u \propto x\,y$ *when both* x *and* y *are variable.*

The variation of u depends upon that of both x and y. Let the variations of x and y take place successively; and when x is changed to x_1, let u be changed to u'; then since $u \propto x$ when y is constant, by § 200 we have

$$\frac{u}{u'} = \frac{x}{x_1}. \qquad (1)$$

Next let y be changed to y_1, and in consequence let u be changed from u' to u_1; then, since $u \propto y$ when x is constant,

$$\frac{u'}{u_1} = \frac{y}{y_1}. \qquad (2)$$

Multiplying (1) by (2), $\dfrac{u}{u_1} = \dfrac{xy}{x_1 y_1}$;

hence by § 200 $u \propto xy$.

This proposition is illustrated by the dependence of the amount of work done, upon the number of men, and the length of time.

Thus, Work $\propto$ time (number of men constant).
Work $\propto$ number of men (time constant).
$\therefore$ Work $\propto$ time $\times$ number of men (when both vary).

The proposition given above can easily be extended to the case in which the variation of u depends upon that of more than two variables. Moreover the variations may be either direct or inverse.

Example. The time of a railway journey varies directly as the distance, and inversely as the velocity; the velocity varies directly as the square root of the quantity of coal used per mile,

and inversely as the number of cars in the train. In a journey of 25 miles in half an hour, with 18 cars, 10 cwt. of coal is required; how much coal will be consumed in a journey of 21 miles in 28 minutes with 16 cars?

Let $t =$ the time in hours,
$d =$ the distance in miles,
$v =$ the velocity in miles per hour,
$n =$ the number of cars,
$q =$ the quantity of coal in cwt.

Then $t \propto \frac{d}{v}$, and $v \propto \frac{\sqrt{q}}{n}$.

$$\therefore \quad t \propto \frac{nd}{\sqrt{q}}, \text{ or } t = a\frac{nd}{\sqrt{q}}. \qquad (1)$$

Now $q = 10$ when $d = 25$, $t = \frac{1}{2}$, and $n = 18$; hence from (1)

$$\frac{1}{2} = a\frac{18 \times 25}{\sqrt{10}}; \therefore a = \frac{\sqrt{10}}{36 \times 25}. \quad \therefore t = \frac{\sqrt{10}}{36 \times 25} \cdot \frac{nd}{\sqrt{q}}.$$

Hence when $d = 21$, $t = \frac{28}{60}$, and $n = 16$, we have

$$\frac{28}{60} = \frac{\sqrt{10} \times 16 \times 21}{25 \times 36\sqrt{q}}; \quad \therefore q = 6\tfrac{2}{5}.$$

Hence the quantity of coal consumed is $6\frac{2}{5}$ cwt.

EXERCISE 17.

1. If $a : b = c : d$, and $b : x = d : y$, prove that $a : x = c : y$.

2. If $a : b = b : c$, prove that $a : c = a^2 : b^2 = b^2 : c^2$.

3. If $a : b = b : c = c : d$, prove that $a : d = a^3 : b^3 = \dots$
   Let $r = a \div b$; then $a = br$, $b = cr$, $c = dr$.
   $\therefore \quad abc = bcdr^3, \quad \therefore a \div d = r^3 = a^3 \div b^3 = \dots$

If $a : b = c : d$, prove that

4. $a^2 c + a c^2 : b^2 d + b d^2 = (a + c)^3 : (b + d)^3.$

5. $a - c : b - d = \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}.$

6. $\sqrt{a^2 + c^2} : \sqrt{b^2 + d^2} = \sqrt{ac + \frac{c^3}{a}} : \sqrt{bd + \frac{d^3}{b}}.$

7. If a, b, c, d, be any four numbers; find what quantity must be added to each to make them proportional.

8. If y varies as x, and $y = 8$ when $x = 15$; find y when $x = 10$.

9. If y varies inversely as x, and $y = 7$ when $x = 3$; find y when $x = 2\frac{1}{3}$.

10. If u varies directly as the square root of x, and inversely as the cube of y, and if $u = 3$ when $x = 256$ and $y = 2$; find x when $u = 24$ and $y = \frac{1}{2}$.

11. If u varies as x and y jointly, while x varies as z^2, and y varies inversely as u; show that u varies as z.

12. The pressure of wind on a plane surface varies jointly as the area of the surface, and the square of the wind's velocity. The pressure on a square foot is 1 lb. when the wind is moving at the rate of 15 miles per hour. Find the velocity of the wind when the pressure on a square yard is 16 lbs.

CHAPTER XI.

THE PROGRESSIONS.

204. An **Arithmetical Progression** is a series of quantities in which each, after the first, equals the preceding plus a *common difference.*

The common difference may be positive or negative. The quantities are called the *terms* of the progression.

205. Let d denote the common difference, a the first term, and l the nth, or last term.

Then by definition

$$\text{the 2d term} = a + d,$$

$$\text{the 3d term} = a + 2\,d,$$

$$\text{and the } n\text{th term} = a + (n - 1)\,d.$$

Hence $$l = a + (n - 1)\,d. \qquad (1)$$

Let S denote the sum of the terms; then

$$S = a + (a + d) + (a + 2\,d) + \cdots + (l - d) + l,$$

$$S = l + (l - d) + (l - 2\,d) + \cdots + (a + d) + a.$$

Adding these two equations, we obtain

$$2\,S = n\,(a + l).$$

Hence $$S = \frac{n}{2}(a + l). \qquad (2)$$

From (1) and (2),

$$S = \frac{n}{2}\{2a + (n - 1)d\}. \qquad (3)$$

If any three of the five quantities, a, l, d, n, S, be given, the other two may be found by the formulas given above.

206. The m terms lying between any two terms of an arithmetical progression (A. P.) are called the m *Arithmetical Means* between the two terms.

207. *To insert* m *arithmetical means between* a *and* b.

Calling a the first term, b will evidently be the $(m + 2)$th term; hence from (1) of § 205,

$$b = a + (m + 1)d.$$

$$\therefore \quad d = \frac{b - a}{m + 1}.$$

Hence the required terms are

$$a + \frac{b - a}{m + 1}, \quad a + \frac{2(b - a)}{m + 1}, \quad \cdots, \quad a + \frac{m(b - a)}{m + 1}.$$

208. COROLLARY. If $m = 1$, the arithmetical mean is $a + \dfrac{b - a}{1 + 1}$, or $\dfrac{a + b}{2}$.

209. If any two terms of an A. P. are given, the progression can be entirely determined; for the data furnish *two* simultaneous equations between the first term and the common difference.

EXAMPLE. The 54th and 4th terms of an A. P. are -61 and 64; find the 27th term.

Here $\quad -61 = \text{the 54th term} = a + 53d;$

and $\quad 64 = \text{the 4th term} = a + 3d.$

Hence $\quad d = -\frac{5}{2}, \qquad a = 71\frac{1}{2}.$

$\therefore$ 27th term $= a + 26d = 6\frac{1}{2}.$

210. A **Geometrical Progression** (G. P.) is a series of quantities in which each, after the first, equals the preceding multiplied by a *constant factor*. The constant factor is called the *ratio* of the progression.

211. Let r denote the ratio, a the first term, l the nth, or last term; then

$$\text{the 2d term} = ar,$$
$$\text{the 3d term} = ar^2.$$
$$\text{and the } n\text{th term} = ar^{n-1}.$$

Hence $$l = ar^{n-1}. \qquad (1)$$

Let S denote the sum of the terms; then

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$
$$= a(1 + r + r^2 + \cdots + r^{n-1}) = \frac{a(r^n - 1)}{r - 1}.$$

Hence $$S = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

If $r < 1$, formula (2) is usually written

$$S = \frac{a(1 - r^n)}{1 - r}. \qquad (3)$$

From (3), $$S = \frac{a}{1 - r} - \frac{a r^n}{1 - r}.$$

Now if $r < 1$, then the greater the value of n, the smaller the value of r^n, and consequently of $\frac{a r^n}{1 - r}$.

Hence, if n be increased without limit, the sum of the progression will approach indefinitely near to $\frac{a}{1 - r}$.

That is, *the limit of the sum of an infinite number of terms of a decreasing G. P. is* $\frac{a}{1 - r}$, or more briefly, *the sum to infinity* is $\frac{a}{1 - r}$.

212. The m terms lying between two terms of a G. P. are called the m *Geometrical Means* between the two terms.

213. *To insert* m *geometrical means between* a *and* b.

Calling a the first term, b will be the $(m + 2)$th term; hence by (1) of § 211.

$$b = a r^{m+1}.$$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{m+1}}. \qquad (1)$$

Hence the required terms are $a r$, $a r^2$, $\cdots$ $a r^m$, in which r has the value in (1).

214. COROLLARY. If $m = 1$, $r = \left(\frac{b}{a}\right)^{\frac{1}{2}}$, and therefore $ar = \sqrt{ab}$; hence *the geometrical mean between* a *and* b *is the mean proportional between* a *and* b.

215. A series of quantities is in **Harmonic Progression** (H. P.) when their reciprocals are in A. P.

Thus, the two series of quantities,

$$1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \cdots, \text{ and } 4, -4, -\tfrac{4}{3}, \cdots,$$

are each in H. P., for their reciprocals,

$$1, 3, 5, 7, \cdots, \text{ and } \tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{3}{4}, \cdots,$$

are in A. P.

We cannot find any general formula for the sum of any number of terms of an H. P. Problems in H. P. are generally solved by inverting the terms and making use of the properties of the resulting A. P.

(1) Continue to 3 terms each way the series 2, 3, 6.

The reciprocals $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ are in A. P.; $\therefore\ d = -\frac{1}{6}$.

$\therefore$ The A. P. is $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, -\frac{1}{3}$.

$\therefore$ The H. P. is $1, \frac{6}{5}, \frac{3}{2}, 2, 3, 6, \infty, -6, -3$.

(2) Insert 4 harmonic means between 2 and 12.

The 4 arithmetical means between $\frac{1}{2}$ and $\frac{1}{12}$ are $\frac{5}{12}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$; hence the harmonic means required are $2\frac{2}{5}$, 3, 4, 6.

216. If H be the harmonic mean between a and b, then by § 215,

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b}, \text{ are in A. P.}$$

$$\therefore\ \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}.$$

$$\therefore\ \frac{2}{H} = \frac{1}{a} + \frac{1}{b}, \text{ or } H = \frac{2ab}{a+b}.$$

217. COROLLARY. If A and G be respectively the arithmetical and the geometrical mean between a and b, then (§ § 208, 214, 216)

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}.$$

$$\therefore\ A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2.$$

Hence $\quad A : G = G : H.$

That is, the geometrical mean between two numbers is also the geometrical mean between the arithmetical and harmonic means of the numbers.

EXERCISE 18.

1. Sum $2, 3\frac{1}{4}, 4\frac{1}{2}, \cdots$, to 20 terms.

2. Sum $\frac{3}{4}, \frac{2}{3}, \frac{7}{12}, \cdots$, to 19 terms.

3. Sum $a - 3b,\ 2a - 5b,\ 3a - 7b, \cdots$, to 40 terms.

4. Sum $2a - b,\ 4a - 3b,\ 6a - 5b,\ \cdots$, to n terms.

5. Insert 17 arithmetical means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

6. The sum of 15 terms of an A. P. is 600, and the common difference is 5; find the first term.

7. How many terms of the series $9, 12, 15, \cdots$, must be taken to make 306?

8. Sum $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \cdots$, to 7 terms.

9. Sum $1, \sqrt{3}, 3, \cdots$, to 12 terms.

10. Insert 5 geometrical means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

11. Sum to infinity $\frac{9}{8}, -1, \frac{8}{9}, \cdots$

12. Sum to infinity $3, \sqrt{3}, 1, \cdots$

13. The 5th and the 2d term of a G. P. are respectively 81 and 24; find the series.

14. The sum of a G. P. is 728, the common ratio 3, and the last term 486; find the first term.

15. The sum of a G. P. is 889, the first term 7, and the last term 448; find the common ratio.

16. Find the 4th term in the series $2, 2\frac{1}{2}, 2\frac{1}{3}, \cdots$

17. Insert four harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

18. Find the two numbers between which 12 and $9\frac{3}{5}$ are respectively the geometrical and the harmonic mean.

19. If a body falling to the earth descends $16\frac{1}{12}$ feet the first second, 3 times as far the next, 5 times as far the third, and so on; how far will it fall during the tth second? How far will it fall in t seconds?

20. A ball falls from the height of 100 feet, and at every fall rebounds one fourth the distance; find the distance passed through by the ball before it comes to rest.

21. According to the law of fall given in Example 19, how long will it be before the ball in Example 20 comes to rest?

Ans. $\frac{60}{193}\sqrt{279} = 5.192+$ seconds.

SECOND PART.

CHAPTER XII.

FUNCTIONS AND THEORY OF LIMITS.

218. A **Variable** is a quantity that is, or is supposed to be, changing in value. Variables are usually represented by the final letters of the alphabet, as x, y, z.

The time since any past event is a variable. The length of a line while it is being traced by a moving point, is a variable. If x represents any variable, x^3, $3x^2$, and $2x^4 - 4x$ will denote variables also.

A **Constant** is a quantity whose value is, or is supposed to be, fixed. Constants are usually represented by figures, or the first letters of the alphabet. Particular values of variables are constants, and they are often denoted by the last letters with accents, as x', y', x'', y''.

The time between any two given dates is a constant, as is also the distance between two fixed points. Figures denote *absolute*, and letters denote *arbitrary* constants.

219. An **Independent Variable** is one whose value does not depend upon any other variable.

A **Dependent Variable** is one whose value depends upon one or more other variables. A dependent variable is called a **Function** of the variable or variables upon which it depends.

If the radius of the base of a cylinder is an independent variable, and the altitude is always four times the radius, then the altitude is a function of the radius, and the surface and volume are different functions of both the radius and the altitude.

The expressions ax^3, $x^4 - cx$, a^x, represent functions of x.

If in any equation between x and y, we regard x as an independent variable, then y is a function of x. Thus, if

$y = 2x^2 + x - 6$, and x increases; then when

$x = -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots$

$y = 22, 9, 0, -5, -6, -3, 4, 15, 30, \cdots$

Here while x increases from -4 to -1, y decreases from 22 to -5; while x increases from -1 to 1, y first decreases and then increases; and while x increases from 1 to 4, y increases from -3 to 30.

220. Functional Notation. The symbol $f(x)$, read "function of x," is used to denote any function of x. When several different functions of x occur in the same discussion, we employ other symbols, as $f'(x)$, $F(x)$, $\phi(x)$, which are read "f prime function of x," "large F function of x," "ϕ function of x," respectively.

The symbols $f(a)$, $f(2)$, $f(z)$, $f(c + d)$, represent the values of $f(x)$ for $x = a$, 2, z, $c + d$, respectively.

Thus, if $f(x) \equiv x^3 + x$, $f(a) \equiv a^3 + a$, $f(2) \equiv 10$,

and $f(c + d) \equiv c^3 + 3c^2d + 3cd^2 + d^3 + c + d$.

Since $f(x)$ denotes any function of $x, y = f(x)$ represents any equation involving x and y, when solved for y.

221. The symbol $\lfloor n$, read "factorial n," denotes the product of the first n whole numbers; that is,

$$\lfloor n \equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n.$$

Thus, $\lfloor 3 \equiv 1 \cdot 2 \cdot 3 \equiv 6$; $\lfloor 4 \equiv 1 \cdot 2 \cdot 3 \cdot 4 \equiv 24$.

EXERCISE 19.

In the first three examples the student should carefully note how $f(x)$ changes as x increases.

1. $f(x) \equiv 5x^2 - 3x + 2$; find $f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), f(4), f(5), f(6)$.

2. $f(x) \equiv 4x^3 - x^4 + 2x - 17$; find $f(-2), f(-1), f(0), f(1), f(2), f(3), f(4), f(5), f(6)$.

3. $f(x) \equiv x^3 + x^2 + 2$; find $f(-4), f(-3), f(-2), f(-1), f(-0.3), f(-0.2), f(0), f(1), f(2)$.

4. $F(x) \equiv x^3 + 3x$; find $F(x_1 + 2), F(x_1 + n), F(5x_1), F(1 \div x_1)$.

5. $F(x) \equiv x^2 + 4x + 3$; find $F(3x_1), F(x_1 + h), F(x_1 - 5), F(c \div a)$.

6. $f(x) \equiv (a + x)^m$; find $f(0), f(1), f(2), f(b), f(z)$.

7. $f(x) \equiv a^x$; find $f(0), f(1), f(2), f(3), f(y), f(m-n)$.

Verify the following identities:

8. $\underline{|5} \equiv 120$; $\underline{|7} \equiv 5040$; $\underline{|9} \equiv 362880$.

9. $\dfrac{\underline{|8}}{\underline{|3}\ \underline{|5}} \equiv 56$; $\dfrac{\underline{|11}}{\underline{|5}\ \underline{|6}} \equiv 462$; $\dfrac{\underline{|9}}{\underline{|8}} \equiv 9$; $\dfrac{\underline{|n}}{\underline{|n-1}} \equiv n$.

10. $9 \cdot 8 \cdot 7 \cdot \underline{|6} \equiv \underline{|9}$;
$n(n-1)(n-2)\cdots(n-r+1)\,\underline{|n-r} \equiv \underline{|n}$.

11. $\dfrac{9 \cdot 8 \cdot 7 \cdot 6}{\underline{|3}} \equiv \dfrac{\underline{|9}}{\underline{|3}\ \underline{|5}}$;
$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} \equiv \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}.$$

THEORY OF LIMITS.

222. Limit of a Variable. When, according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the **Limit** of the variable. A variable may be always less, always greater, or alternately less and greater than its limit.

If a regular polygon be inscribed in a circle, and another be circumscribed about it, and if the number of their sides be doubled again and again, the area of each of these polygons will approach indefinitely near to, but can never equal, the area of the circle, which is therefore their common limit. The area of the inscribed polygon is always less than its limit, while that of the circumscribed is always greater.

The limit of the perimeters of each of these polygons is evidently the circumference of the circle. The variable difference

between the area of the circle and that of either polygon continually decreases, and evidently approaches zero as its limit.

If n increases without limit $\left(\frac{1}{2}\right)^n$, or $\frac{1}{2^n}$, approaches zero as its limit; for by increasing n, $\frac{1}{2^n}$ can be made as small as we please, but it cannot be made zero.

223. COROLLARY 1. The difference between a variable and its limit is a variable whose limit is zero; that is, if a is the limit of x, the limit of $a-x$ is zero.

When near its limit, a variable whose limit is zero is called an *Infinitesimal.*

224. COROLLARY 2. A variable cannot approach two unequal limits at the same time.

For if so, in approaching one of these limits the variable would evidently reach a value intermediate between the two unequal limits, and then it would recede from one of them while it approached the other.

225. COROLLARY 3. If the limit of v is zero, the limit of cv is zero also, and therefore the limit of $ca - cv$ is ca, c and a being any constants.

For however small k may be, we may make $v<k \div c$, whence $cv < k$; that is, cv can be made as small as we please, but it cannot be made zero, since v cannot; hence the limit of cv is zero.

Again, $ca - cv$ evidently approaches as near to ca as cv does to zero; hence the limit of $ca - cv$ is ca.

Notation. The sign, $\doteq$, denotes "approaches as a limit;" thus, $x \doteq a$ is read "x approaches a as its limit."

The limit of x is often written briefly lt (x).

226. *If two variables are always equal and one approaches a limit, the other approaches the same limit; that is, if* $y = x$, *and* $x \doteq a$, *then* $y \doteq a$.

For evidently if x approaches indefinitely near to a, but cannot reach it, then, since $y = x$, y also must approach indefinitely near to a, but cannot reach it.

227. *If two variables are always equal and each approaches a limit, their limits are equal; that is, if* $y = x$, *and* $x \doteq a$, *and* $y \doteq b$, *then* $b = a$.

If $y = x$, and $x \doteq a$, then, by § 226, $y \doteq a$. But $y \doteq b$, whence $b = a$, since, by § 224, y cannot approach two unequal limits at the same time.

228. *The limit of the product of a constant and a variable is the product of the constant and the limit of the variable; that is, if lt* $(x) = a$, *lt* $(c\,x) = c\,a$.

Let $\qquad v = a - x;$

then $\qquad \text{lt}\,(v) = \text{lt}\,(a - x) = 0, \qquad$ §§ 227, 223.

and $\qquad c\,x = c\,a - c\,v.$

Hence $\qquad \text{lt}\,(c\,x) = \text{lt}\,(c\,a - c\,v) = c\,a. \qquad$ §§ 227, 225.

229. *The limit of the variable sum of a finite number of variables is the sum of their limits; that is, if* $lt\,(x) = a$, $lt\,(y) = b$, $lt\,(z) = c$, *etc. to* n *variables; then*

$$\text{lt}\,(x + y + z + \cdots) = a + b + c + \cdots$$

For let $v_1 = a - x,\ v_2 = b - y,\ v_3 = c - z, \cdots;$

then $\text{lt}\,(v_1) = 0,\ \text{lt}\,(v_2) = 0,\ \text{lt}\,(v_3) = 0, \cdots,$

and $x + y + z + \cdots = (a + b + c + \cdots) - (v_1 + v_2 + v_3 + \cdots).$

Hence

$\text{lt}\,(x + y + z + \cdots) = \text{lt}[(a + b + c + \cdots) - (v_1 + v_2 + v_3 + \cdots)].$

Now however small k may be, each one of the n variables, $v_1, v_2, v_3, \cdots$, can be made less than $k \div n$; therefore their sum can be made less than k; hence

$$\text{lt}\,(v_1 + v_2 + v_3 + \cdots) = 0.$$

Hence $\text{lt}\,(x + y + z + \cdots) = a + b + c + \cdots$

230. *The limit of the variable product of two or more variables is the product of their limits; that is, if* $lt\,(x) = a$, *and* $lt\,(y) = b$, *then*

$$\text{lt}\,(xy) = ab = \text{lt}\,(x)\ \text{lt}\,(y).$$

Let $v_1 = a - x$ and $v_2 = b - y;$

then $x = a - v_1,\ y = b - v_2,$

and $xy = ab - av_2 - bv_1 + v_1 v_2.$

$$\therefore \text{lt}\,(xy) = \text{lt}\,(ab - av_2) - \text{lt}\,(bv_1) + \text{lt}\,(v_1 v_2) \quad \S\ 229.$$
$$= ab = \text{lt}\,(x)\ \text{lt}\,(y).$$

In like manner the theorem is proved for n variables.

231. *The limit of the variable quotient of two variables is the quotient of their limits; that is,*

$$\text{lt}\,(x \div y) = \text{lt}\,(x) \div \text{lt}\,(y).$$

Let $z = x \div y$, or $x = yz$;

then $\text{lt}\,(z) = \text{lt}\,(x \div y)$,

and $\text{lt}\,(x) = \text{lt}\,(yz) = \text{lt}\,(y)\,\text{lt}\,(z)$. (1)

Hence $\text{lt}\,(z) = \text{lt}\,(x) \div \text{lt}\,(y)$. (2)

Whence $\text{lt}\,(x \div y) = \text{lt}\,(x) \div \text{lt}\,(y)$.

REMARK. The demonstration given above fails, and the theorem is not true, when $\text{lt}\,(y)$, or the limit of the divisor, is zero; for then we cannot divide (1) by $\text{lt}\,(y)$ to obtain (2).

232. *When the product, quotient, or sum of two or more variables is equal to a constant, the product, quotient, or sum of their limits, is equal to the same constant.*

(i.) Let $xy = m$; then $xyz = mz$.

$$\therefore\ \text{lt}\,(x)\,\text{lt}\,(y)\,\text{lt}\,(z) = m\,\text{lt}\,(z).$$

$$\therefore\ \text{lt}\,(x)\,\text{lt}\,(y) = m.$$

(ii.) Let $x \div y = m$; then $x = my$.

$$\therefore\ \text{lt}\,(x) = m\,\text{lt}\,(y), \text{ or } \text{lt}\,(x) \div \text{lt}\,(y) = m.$$

(iii.) Let $x + y + z + \cdots = m$;

then $y + z + \cdots = m - x$.

$$\therefore\ \text{lt}\,(y) + \text{lt}\,(z) + \cdots = m - \text{lt}\,(x).$$

$$\therefore\ \text{lt}\,(x) + \text{lt}\,(y) + \text{lt}\,(z) + \cdots = m.$$

233. If a is finite, and $x \doteq 0$, then the fraction $\frac{a}{x}$ will numerically ***increase without limit.***

If x increases without limit, then $\frac{a}{x} \doteq 0$.

A variable that increases without limit is called an **Infinite.** The value of an infinite is denoted by the symbol ∞, and $x = \infty$ is read "x increases without limit," or "x is infinite." With this notation the two statements made above may be written as follows:

$$\text{If } x \doteq 0, \text{ then } \frac{a}{x} = \infty;$$

$$\text{if } x = \infty, \text{ then } \frac{a}{x} \doteq 0.$$

EXAMPLE 1. Find lt $\frac{3x^3-2x^2-4}{5x^3-4x+8}$, if $x = \infty$.

$$\text{If } x = \infty, \text{ lt}\left(\frac{3x^3-2x^2-4}{5x^3-4x+8}\right) = \text{lt}\left(\frac{3-\frac{2}{x}-\frac{4}{x^3}}{5-\frac{4}{x^2}+\frac{8}{x^3}}\right) = \frac{3}{5}.$$

EXAMPLE 2. If $x \doteq 0$, and $a > 0$, then lt$(a^x) = 1$.

Let $a > 1$, x positive, and k a positive number as small as we please; then, as $1 \div x = \infty$,

$$(1+k)^{\frac{1}{x}} > a > 1;$$

$$\therefore \quad 1 + k > a^x > 1, \text{ or lt}(a^x) = 1.$$

The proof is similar for $a < 1$. For x negative we have $a^x = (1 \div a)^{-x}$, in which $-x$ is positive.

In this example x is commensurable, and only the positive real values of a^x are considered.

EXERCISE 20.

1. Prove $\text{lt}\,(x^n) = [\text{lt}\,(x)]^n$.

$$\begin{aligned}\text{Lt}\,(x^n) &= \text{lt}\,(x \cdot x \cdots \text{to } n \text{ factors})\\ &= \text{lt}\,(x) \cdot \text{lt}\,(x) \cdots \text{to } n \text{ factors}\\ &= [\text{lt}\,(x)]^n.\end{aligned}$$

2. Prove $\text{lt}\left(x^{\frac{m}{n}}\right) = [\text{lt}\,(x)]^{\frac{m}{n}}$.

Let $x^{\frac{m}{n}} = z$; then $x^m = z^n$, etc.

3. Prove $\text{lt}\left(\dfrac{m}{x^2}\right) = m \div [\text{lt}\,(x)]^2$.

If $\text{lt}\,(x) = a$, $\text{lt}\,(y) = b$, $\text{lt}\,(z) = c$, and $\text{lt}\,(v) = 0$, find

4. $\text{Lt}\,(xyz + axz)$.

5. $\text{Lt}\left(x^{\frac{1}{2}}y^{\frac{3}{4}} + mxz^3 + nxzv\right)$.

6. $\text{Lt}\left(\dfrac{x^2y^{\frac{1}{2}} + mz^3 + nzv}{xy + nx^{\frac{3}{2}} + myv}\right)$.

7. $\text{Lt}\left(\dfrac{xyv}{z} + \dfrac{m\sqrt{xy}}{r\sqrt[3]{xz}}\right)$.

Find the limit of each of the following expressions, (i.) when $x \doteq 0$, (ii.) when $x = \infty$:

8. $\dfrac{8x^2 + 2x}{2x^2 + 4}$.

9. $\dfrac{ax^3 + bx^2 + cx + e}{mx^3 + nx^2 + px + q}$.

10. $\dfrac{(2x - 3)(3 - 5x)}{7x^2 - 6x + 4}$.

11. $\dfrac{(3x^2 - 1)^2}{x^4 + 9}$.

12. $\dfrac{(3 + 2x^3)(x - 5)}{(4x^3 - 9)(1 + x)}$.

13. $\dfrac{1 - x^2}{2x^3 - 1} \div \dfrac{1 - x}{2x^2}$.

234. Vanishing Fractions. If in the fraction $\frac{x^2+ax-2a^2}{x^2-a^2}$ we put $x=a$, it will assume the indeterminate form $\frac{0}{0}$. A fraction which assumes this form for any particular value of x, as a, is called a **Vanishing Fraction** for $x=a$.

To find the value of such a fraction for $x=a$, we find its limit when $x \doteq a$.

The limit of $\frac{x^2+ax-2a^2}{x^2-a^2}$ when $x \doteq a$ is often written $\underset{x \doteq a}{\text{limit}} \left[\frac{x^2+ax-2a^2}{x^2-a^2}\right]$.

EXAMPLE. Find $\underset{x \doteq a}{\text{limit}} \left[\frac{x^2+ax-2a^2}{x^2-a^2}\right]$.

Here the limit of the divisor, x^2-a^2, is zero, and we cannot apply § 231. But as long as x is not absolutely equal to a, we may divide both terms of the fraction by $x-a$; hence

$$\frac{x^2+ax-2a^2}{x^2-a^2} = \frac{x+2a}{x+a}.$$

$$\therefore \underset{x \doteq a}{\text{limit}} \left[\frac{x^2+ax-2a^2}{x^2-a^2}\right] = \underset{x \doteq a}{\text{limit}} \left[\frac{x+2a}{x+a}\right] = \frac{3}{2}.$$

235. Incommensurable Exponents. *If* a *is positive and* m *is incommensurable, the positive real value of* a^m *is the limit of the positive real value of* a^x *when* $x \doteq m$.

Let x and y be commensurable, and let

$$x < m < y, \ \text{lt}(x) = m = \text{lt}(y).$$

Then, if $a > 1$, we assume as axiomatic that

$$a^m < a^y, \text{ or } a^m - a^x < a^y - a^x.$$

But $\quad a^y - a^x \equiv a^x(a^{y-x} - 1) \doteq 0.$ Example 2, § 233.

$$\therefore \quad a^m = \text{lt}(a^x) \text{ when } a > 1 \text{ and } x \doteq m.$$

The proof is similar when $a < 1$.

This proof applies also when m is commensurable.

EXAMPLE. Prove the laws in § 70, when m and n are incommensurable, a and b being positive.

Let x and y denote any two commensurable fractions whose limits are m and n respectively; then

$$\text{lt}(a^x a^y) = \text{lt}(a^{x+y}) = a^{m+n},$$

also $\quad \text{lt}(a^x a^y) = \text{lt}(a^x) \cdot \text{lt}(a^y) = a^m a^n.$

$$\therefore \quad a^m a^n = a^{m+n}.$$

Similarly the other laws are proved.

REMARK. Except when the limit of a divisor is zero, the limit of each function considered in this chapter is the result obtained by substituting for the variables their respective limits.

EXERCISE 21.

Find

1. $\underset{x \doteq 1}{\text{Limit}} \left[\dfrac{x^3 - 1}{x - 1}\right].$

2. $\underset{x \doteq -1}{\text{Limit}} \left[\dfrac{x^3 + 1}{x^2 - 1}\right].$

3. $\underset{x \doteq a}{\text{Limit}} \left[\dfrac{(x^2 - a^2)^{\frac{1}{3}}}{(x - a)^{\frac{1}{3}}}\right].$

4. $\underset{x \doteq a}{\text{Limit}} \left[\dfrac{x^4 - a^4}{x - a}\right].$

5. $\underset{x \doteq a}{\text{Limit}} \left[\dfrac{x^5 - a^5}{x - a}\right].$

6. $\underset{x \doteq a}{\text{Limit}} \left[\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{a - x}}\right].$

In Example 6 rationalize the numerator.

CHAPTER XIII.

DIFFERENTIATION.

236. The amount of any change (increase or decrease) in the value of a variable is called an **Increment.** If a variable is increasing, its increment is *positive;* if it is decreasing, its increment is *negative.* An increment of a variable is denoted by writing the letter Δ before it; thus Δx, Δy, Δz, denote the increments of x, y, z, respectively. Hence if x' denote any value of x, $x' + \Delta x$ denotes a subsequent value of x. If y is a function of x, and $y = y'$ when $x = x'$; then $y = y' + \Delta y$, when $x = x' + \Delta x$.

237. The **Derivative** of a function is the limit of the ratio of the increment of the function to the increment of the variable as the increment of the variable approaches zero as its limit.

If y is a function of x, the derivative of y with respect to x is often denoted by $D_x y$. Hence by definition, we have

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = D_x y. \qquad [1]$$

EXAMPLE 1. Given $y = a x^2 + c x + b$, to find $D_x y$.

Let x' and y' be any two corresponding values of x and y; then

$$y' = a x'^2 + c x' + b. \qquad (1)$$

When $x = x' + \Delta x$, $y = y' + \Delta y$; hence we have

$$y' + \Delta y = a (x' + \Delta x)^2 + c (x' + \Delta x) + b$$
$$= a x'^2 + 2 a x' \Delta x + a \Delta x^2 + c x' + c \Delta x + b. \qquad (2)$$

Subtracting (1) from (2), we obtain

$$\Delta y = 2 a x' \Delta x + a \Delta x^2 + c \Delta x.$$

Dividing by Δx, we have

$$\frac{\Delta y}{\Delta x} = 2 a x' + a \Delta x + c.$$

$$\therefore \quad \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} [2 a x' + a \Delta x + c] \qquad \S\ 227.$$
$$= 2 a x' + c.$$

Hence, as x' is any value of x, we have in general,

$$D_x y = 2 a x + c.$$

To the expression $\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right]$ we cannot apply the principle of § 231, for the limit of the divisor, Δx, is zero.

EXAMPLE 2. Given $y = m x + b$, to find $D_x y$.

Here we find that $\frac{\Delta y}{\Delta x} = m$; hence, $\frac{\Delta y}{\Delta x}$, being constant, cannot approach a limit, as $\Delta x \doteq 0$. In this case, $D_x y$ is the constant ratio of Δy to Δx; that is, $D_x y = m$. It is evident that the ratio $\Delta y \div \Delta x$ will be constant, as $\Delta x \doteq 0$, only when the given equation is of the first degree in x and y.

238. *The derivative of a function is positive or negative according as the function increases or decreases, when its variable increases; and conversely.*

For if y increases when x increases, Δy and Δx have like signs; hence, from [1] of § 237, $D_x y$ is positive. If y decreases when x increases, Δy and Δx have unlike signs; hence $D_x y$ is negative; and conversely.

EXERCISE 22.

Find the derivative of y, if

1. $y = x^2.$
2. $y = ax^2 + b.$
3. $y = cx^2 - ax.$
4. $y = x^3.$
5. $y = cx^3 - ax^2.$
6. $y = x^3 - cx^2 + 4x.$
7. $y = ax^4.$
8. $y = cx^4 - bx^2.$
9. $y = a \div x.$
10. $y = x \div (a - x).$

239. Differentials. Any increment of an independent variable may be taken as its **Differential.**

The **Differential** of any *function* is the product of its derivative into the differential of its variable. The differentials of x and y are denoted by dx and dy respectively. Hence, if y is a function of x, by definition we have

$$dy = D_x y\, dx. \qquad (1)$$

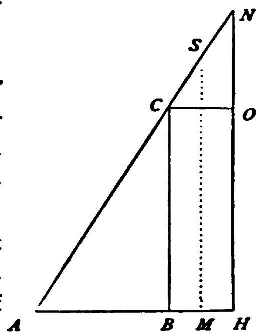
Dividing both members of (1) by dx, we obtain

$$\frac{dy}{dx} = D_x y. \qquad (2)$$

From (2), and [1] of § 237, we have

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}. \qquad [2]$$

The following geometrical illustration of increments, differentials, and a derivative may aid the student.



Conceive a variable right triangle as generated by the perpendicular BC moving uniformly to the right. Let x denote the variable base, $2ax$ its altitude, and y its area; then, by geometry, $y = ax^2$.

Let AB be any value of x, and let $\Delta x = BH$; then $\Delta y =$ area $BHNC$.

Let MS join the middle points of BH and CN; then, by geometry,

$$\frac{\Delta y}{\Delta x} = \frac{\text{area } BHNC}{BH} = MS. \qquad (1)$$

Now, as $\Delta x \doteq 0$, $MS \doteq BC$; hence from (1), we have

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}[MS] = BC;$$

$\therefore \qquad D_x y = BC.$

Let $\qquad dx = BH;$

then $\qquad dy = D_x y\, dx = BC \cdot BH =$ area $BHOC$.*

Here $\qquad \Delta y = dy +$ triangle OCN.

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* This illustration shows that the differentials of a function and its variable are what *would* be their simultaneous increments, if at the values considered their change became uniform.

It also illustrates the fact that the derivative of a function equals the ratio of the rate of change of the function to that of its variable.

240. Differentiation is the operation of finding the differential of a function. The sign of differentiation is the letter d; thus d in $d(x^3)$ indicates the *operation* of differentiating x^3, while the whole expression $d(x^3)$ denotes the differential of x^3.

We could differentiate any function by finding its derivative as in § 237; but in practice it is more expedient to use the following general principles:

241. *The differentials of equal functions of the same variable are equal.*

If u and y are equal functions of x, we are to prove that $du = dy$.

If $$u = y,\ \Delta u = \Delta y.$$

Hence $$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]. \qquad \text{§ 227.}$$

Whence $$\frac{du}{dx} = \frac{dy}{dx}, \text{ or } du = dy. \qquad \text{§ 239.}$$

242. *The differential of the product of a constant and a variable is the product of the constant and the differential of the variable.*

We are to prove that $d(ay) = a\,dy$, in which y is some function of x.

Let $u = ay$, and let x' denote any value of x, and y' and u' the corresponding values of y and u respectively; then

$$u' = ay'. \qquad (1)$$

When $x = x' + \Delta x$, then $y = y' + \Delta y$, and $u = u' + \Delta u$; hence

$$u' + \Delta u = a(y' + \Delta y) = a y' + a \Delta y. \qquad (2)$$

Subtracting (1) from (2), we have

$$\Delta u = a \Delta y.$$

$$\therefore \quad \frac{\Delta u}{\Delta x} = a \frac{\Delta y}{\Delta x}.$$

$$\therefore \quad \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[a \frac{\Delta y}{\Delta x}\right] \qquad \S\ 227.$$

$$= a \cdot \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right]. \qquad \S\ 228.$$

$$\therefore \quad \frac{du}{dx} = a \frac{dy}{dx}. \qquad \S\ 239.$$

Hence, as x' is any value of x, we have in general,

$$du = d(ay) = a\,dy.$$

243. *The differential of a constant is zero.*

If a is a constant, $\Delta a = 0$; hence $da = 0$.

244. *The differential of a polynomial is the algebraic sum of the differentials of its several terms.*

We are to prove that $d(v + y - z + a) = dv + dy - dz$, in which v, y, and z are functions of x.

Let $u = v + y - z + a$, and let x' represent any value of x, and v', y', z', and u' the corresponding values of v, y, z, and u, respectively; then

$$u' = v' + y' - z' + a. \qquad (1)$$

When $x = x' + \Delta x$, then $v = v' + \Delta v$, $y = y' + \Delta y$, $z = z' + \Delta z$, and $u = u' + \Delta u$; hence

$$u' + \Delta u = v' + \Delta v + y' + \Delta y - (z' + \Delta z) + a. \quad (2)$$

Subtracting (1) from (2), we have

$$\Delta u = \Delta v + \Delta y - \Delta z.$$

$$\therefore \frac{\Delta u}{\Delta x} = \frac{\Delta v}{\Delta x} + \frac{\Delta y}{\Delta x} - \frac{\Delta z}{\Delta x}.$$

$$\therefore \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta v}{\Delta x} + \frac{\Delta y}{\Delta x} - \frac{\Delta z}{\Delta x}\right]. \quad \S\ 227.$$

$$\therefore \frac{du}{dx} = \frac{dv}{dx} + \frac{dy}{dx} - \frac{dz}{dx}. \quad \S\S\ 229, 239.$$

Hence, as x' is any value of x, we have in general

$$du = d(v + y - z + a) = dv + dy - dz.$$

245. *The differential of the product of two variables is the first into the differential of the second, plus the second into the differential of the first.*

We are to prove that $d(yz) = y\,dz + z\,dy$, in which y and z are functions of x.

Let $u = yz$, and let x' represent any value of x and y', z', and u' the corresponding values of y, z, and u, respectively; then

$$u' = y'z'. \quad (1)$$

When $x = x' + \Delta x$, then $y = y' + \Delta y$, $z = z' + \Delta z$, and $u = u' + \Delta u$; hence

$$u' + \Delta u = (y' + \Delta y)(z' + \Delta z) \qquad (2)$$
$$= y' z' + y' \Delta z + z' \Delta y + \Delta z \Delta y.$$

Subtracting (1) from (2), we obtain

$$\Delta u = y' \Delta z + z' \Delta y + \Delta z \Delta y.$$

$$\therefore \; \frac{\Delta u}{\Delta x} = y' \frac{\Delta z}{\Delta x} + (z' + \Delta z) \frac{\Delta y}{\Delta x}.$$

$$\therefore \; \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta u}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[y' \frac{\Delta z}{\Delta x}\right] + \underset{\Delta x \doteq 0}{\text{limit}} \left[(z' + \Delta z) \frac{\Delta y}{\Delta x}\right].$$

$$\therefore \; \frac{du}{dx} = y' \frac{dz}{dx} + z' \frac{dy}{dx}.$$

Hence, as x' is any value of x, we have in general

$$du = d(yz) = y\,dz + z\,dy.$$

246. *The differential of the product of any number of variables is the sum of the products of the differential of each into all the rest.*

We are to prove that $d(vyz) = yz\,dv + vz\,dy + vy\,dz$, in which v, y, and z are functions of x.

Let $\qquad u = vy;$

then $\qquad d(vyz) = d(uz) \qquad$ § 241.

$$= z\,du + u\,dz \qquad \text{§ 245.}$$
$$= z\,d(vy) + vy\,dz$$
$$= yz\,dv + vz\,dy + vy\,dz. \qquad \text{§ 245.}$$

In a similar manner the theorem may be demonstrated for any number of variables.

247. *The differential of a fraction is the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

We are to prove that $d\left(\frac{y}{z}\right) = \frac{z\,dy - y\,dz}{z^2}$, in which y and z are functions of x.

Let $u = \frac{y}{z}$; then $uz = y$.

$$\therefore\ u\,dz + z\,du = dy. \qquad \text{§ 245.}$$

$$\therefore\ du = \frac{dy - u\,dz}{z} = \frac{dy - \frac{y}{z}\,dz}{z} = \frac{z\,dy - y\,dz}{z^2}.$$

248. By § 247, $d\left(\frac{a}{z}\right) = \frac{z\,da - a\,dz}{z^2} = -\frac{a\,dz}{z^2}$.
($z\,da = 0$, since $da = 0$.)

Hence, *the differential of a fraction with a constant numerator is minus the numerator into the differential of the denominator divided by the square of the denominator.*

249. By § 242, $d\left(\frac{y}{a}\right) = d\left(\frac{1}{a}y\right) = \frac{1}{a}\,dy = \frac{dy}{a}$.

250. *The differential of a variable affected with any constant exponent is the product of the exponent, the variable with its exponent diminished by one, and the differential of the variable.*

(i.) *If the exponent is a positive integer, as* m;

then $d(z^m) = d(z \cdot z \cdot z \cdots \text{ to } m \text{ factors})$

$$= z^{m-1}\,dz + z^{m-1}\,dz + \cdots \text{ to } m \text{ terms}$$

$$= m\,z^{m-1}\,dz.$$

(ii.) *If the exponent is any positive fraction, as* $\frac{m}{n}$;

let $u = z^{\frac{m}{n}}$; then $u^n = z^m$.

$$\therefore\ n\,u^{n-1}\,du = m\,z^{m-1}\,dz.$$

$$\therefore\ du = \frac{m}{n}\,\frac{z^{m-1}}{u^{n-1}}\,dz = \frac{m}{n}\,\frac{z^{m-1}\,u}{u^n}\,dz = \frac{m}{n}\,\frac{z^{m-1}\,z^{\frac{m}{n}}}{z^m}\,dz$$

$$= \frac{m}{n}\,z^{\frac{m}{n}-1}\,dz.$$

(iii.) *If the exponent is any negative quantity, as* —n;

then $$z^{-n} = \frac{1}{z^n}. \qquad (1)$$

Differentiating (1) by § 248, we obtain

$$d(z^{-n}) = \frac{-n\,z^{n-1}}{z^{2n}}\,dz = -n\,z^{-n-1}\,dz.$$

251. By § 250, $d(\sqrt{z}) = d\left(z^{\frac{1}{2}}\right) = \frac{1}{2}\,z^{-\frac{1}{2}}\,dz = \frac{dz}{2\sqrt{z}}$.

Hence, *the differential of the square root of a variable is the differential of the variable divided by twice the square root of the variable.*

252. The general symbol for the derivative of $f(x)$ is $f'(x)$; hence

$$d[f(x)] = f'(x)\,dx.$$

EXERCISE 23.

Differentiate

1. $x^3 + 8x + 2x^2$.

$$d(x^3 + 8x + 2x^2) \equiv d(x^3) + d(8x) + d(2x^2) \qquad \S\ 244.$$
$$\equiv 3x^2dx + 8dx + 4xdx \qquad \S\ 250.$$
$$\equiv (3x^2 + 8 + 4x)dx.$$

2. $y = 3ax^2 - 5nx - 8m$.

$$dy = d(3ax^2 - 5nx - 8m) = d(3ax^2) - d(5nx) - d(8m)$$
$$= (6ax - 5n)dx.$$

3. $f(x) \equiv 5ax^2 - 3b^2x^3 - abx^4$.

$$f'(x)dx \equiv (10ax - 9b^2x^2 - \qquad x^3)dx.$$

4. $f(x) \equiv a^3 + 5b^2x^3 + 7a^3x^5$.

5. $y = ax^{\frac{3}{2}} + bx^{\frac{1}{2}} + c$.

6. $f(x) \equiv (b + ax^2)^{\frac{5}{4}}$.

7. $y = (1 + 2x^2)(1 + 4x^3)$.

8. $y = \dfrac{x + a^2}{x + b}$. 9. $f(x) \equiv \dfrac{a}{b - 2x^2}$.

In Example 8

$$dy = \frac{(x + b)\,d(x + a^2) - (x + a^2)\,d(x + b)}{(x + b)^2} = \frac{b - a^2}{(x + b)^2}dx.$$

10. $y = (a + x)\sqrt{a - x}$. 11. $f(x) \equiv \dfrac{2x^4}{a^2 - x^2}$.

12. $f(x) \equiv \left(\frac{x}{1-x}\right)^m$.

13. $f(x) \equiv \sqrt{\frac{1+x}{1-x}}$.

14. $f(x) \equiv \frac{x^3}{(1-x^2)^{\frac{3}{2}}}$.

15. $y = \frac{x}{\sqrt{1+x^2}}$.

16. $f(x) \equiv \frac{2x^2-3}{4x+x^2}$.

17. $y = \frac{x^3}{a^2-x^2}$.

18. $f(x) \equiv \frac{a}{(b^2+x^2)^3}$.

19. $f(x) \equiv \frac{x^3}{(1+x)^2}$.

20. $f(x) \equiv \sqrt{ax} + \sqrt{c^2x^3}$.

21. $y^2 = 2px$.

In Example 21, $d(y^2) = d(2px)$, etc.

22. $a^2y^2 + b^2x^2 = a^2b^2$.

23. $2xy^2 - ay^2 = x^3$.

24. Prove the theorems of §§ 246, 247 by the general method used in the previous articles.

25. Assuming the binomial theorem, prove the theorem of § 250 by the general method.

253. Successive Derivatives. Since $f'(x)$, the derivative of $f(x)$, is in general another function of x, it can be differentiated the same as $f(x)$. The derivative of $f'(x)$ is called the *Second Derivative* of the original function $f(x)$, and is denoted by $f''(x)$. The derivative of $f''(x)$ is called the *Third Derivative* of $f(x)$, and is represented by $f'''(x)$; and so on. $f''(x)$

represents the nth derivative of $f(x)$, or the derivative of $f^{n-1}(x)$.

Thus, if $f(x) \equiv x^4$, then $f'(x) \equiv 4x^3$,

$$f''(x) \equiv 12x^2, \qquad f'''(x) \equiv 24x,$$

$$f^{\text{IV}}(x) \equiv 24, \qquad f^{\text{V}}(x) \equiv 0.$$

$f'(x), f''(x), f'''(x), \ldots, f^n(x)$ are called the *Successive Derivatives* of $f(x)$.

EXERCISE 24.

Find the successive derivatives of $f(x)$, when

1. $f(x) \equiv x^3 + 2x^2 + x + 7.$
2. $f(x) \equiv cx^3 + ax^2 + b.$
3. $f(x) \equiv (a + x)^3.$
4. $f(x) \equiv (a + x)^m.$
5. $f(x) \equiv A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6.$

254. Continuity. A variable is **Continuous**, or *varies continuously*, when in passing from one value to another it passes successively through all intermediate values. Otherwise it is *discontinuous*.

A **Continuous** function is one that varies continuously, when its variable is continuous. Hence y is a continuous function of x, if for each real finite value of x, y is real, finite, and determinate, and if $\Delta y \doteq 0$ when $\Delta x \doteq 0$.

The time since any past event is a continuous variable, as is also the length of a line while being traced by a moving point.

The velocity acquired by a falling body, and the distance fallen, are continuous functions of the time.

The area and the altitude of the triangle in § 239 are continuous functions of the base.

The number of sides of a regular polygon inscribed in a circle, when indefinitely increased, is a discontinuous variable, as is also the perimeter or the area of the polygon. For each of these variables passes from one value to another without passing through all intermediate values.

In general $1 \div x$ is a continuous function of x; but when x increases and passes through zero, $1 \div x$ leaps from $-\infty$ to $+\infty$; hence $1 \div x$ is discontinuous for $x = 0$.

255. *Any rational integral function of x is continuous.*

Let n be a positive integer, and

$$y = A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n,$$

then for each real finite value of x, y has one real finite value, and only one.

Again, if by the method in Example 1 of § 237, we obtain Δy in terms of Δx, Δy will be found equal to Δx multiplied by a finite quantity; hence $\Delta y \doteq 0$ when $\Delta x \doteq 0$.

Therefore y, or $A_0 x^n + A_1 x^{n-1} + \cdots + A_n$, is a continuous function of x.

Thus, if $y = x^3 + 2x^2 + x$, (1)

$$\Delta y = (3x^2 + 3x\Delta x + \Delta x^2 + 2x + \Delta x + 1)\Delta x. \quad (2)$$

From (1), y has one finite real value for each finite real value of x, and from (2), $\Delta y \doteq 0$ when $\Delta x \doteq 0$. Hence $x^3 + 2x^2 + x$ is a continuous function of x.

CHAPTER XIV.

DEVELOPMENT OF FUNCTIONS IN SERIES.

256. A **Series** is an expression in which the successive terms are formed by some regular law. A **Finite** series is one of which the number of terms is limited. An **Infinite** series is one of which the number of terms is unlimited.

257. An infinite series is said to be **Convergent** when the sum of the first n terms approaches a limit as n is increased indefinitely; and the limit is called the **Sum** of the infinite series. If the sum of the first n terms of an infinite series does not approach a definite limit when $n = \infty$, the series is *Divergent*, and has no sum.

Thus, the infinite geometrical series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

is convergent; for by § 211 the sum of its first n terms approaches $1 \div (1 - \frac{1}{2})$, or 2, as its limit, when $n = \infty$.

The series $1 + 1 + 1 + 1 + \cdots$ is divergent, since the sum of its first n terms increases indefinitely with n.

The series $1 - 1 + 1 - 1 + \cdots$ is divergent; for the sum of its first n terms does not approach a limit, but alternates between 1 and 0 according as n is odd or even.

258. To **Develop** a function is to find a series the sum of which is equal to the function. Hence the development of a function is either a *finite* or a *convergent infinite* series.

Thus, the development of the function $(a+x)^4$ is the finite series $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.

259. Development of Functions by Division.

EXAMPLE 1. Develop $\dfrac{1-x^n}{1-x}$ by division.

Dividing $1-x^n$ by $1-x$, we obtain

$$\frac{1-x^n}{1-x} \equiv 1 + x + x^2 + x^3 + \dots x^{n-1}. \qquad (1)$$

If n is finite, the series in identity (1) is finite and is the development of the function $\dfrac{1-x^n}{1-x}$ for all values of x.

EXAMPLE 2. Develop $\dfrac{1}{1-x}$ by division.

Dividing 1 by $1-x$, we obtain

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^{n-1} + \dots, \qquad (1)$$

in which x^{n-1} is the nth or general term of the series.

The series in (1) is infinite, and hence it must be convergent to be the development of the function.

If x is numerically less than 1, the series is evidently a decreasing geometrical series, of which the first term is 1, and the ratio x; hence by § 211 the sum of n terms approaches $\dfrac{1}{1-x}$ as a limit when $n = \infty$.

If x is numerically greater than 1, the sum increases indefinitely with n; hence the series is divergent, and is not the development of the function. Thus, for $x = 2$, the series becomes

$$1 + 2 + 4 + 8 + 16 + \cdots,$$

while the function equals -1.

If $x = 1$, the sum of the first n terms is n, and therefore the series is divergent.

If $x = -1$, the series becomes the divergent series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Hence the series in (1) is the development of $\frac{1}{1-x}$ only for values of x between -1 and $+1$.

EXAMPLE 3. Develop $\frac{x}{1+x}$ by division.

Dividing x by $1 + x$, we obtain

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5 - \cdots + (-1)^{n-1}x^n + \cdots$$

Here the series is evidently divergent for all values of x except those between -1 and $+1$. For values of x between -1 and $+1$ the series is a decreasing geometrical progression of which the first term is x and the ratio is $-x$; hence the sum of n terms approaches $\frac{x}{1+x}$ as its limit, when $n = \infty$.

PRINCIPLES OF UNDETERMINED COEFFICIENTS.

260. Undetermined Coefficients are assumed coefficients whose values, not known at the outset, are to be determined in the course of the demonstration of a theorem or the solution of a problem.

261. *If* $A_0 \equiv B_0$, $A_1 \equiv B_1$, $A_2 \equiv B_2$, $A_3 \equiv B_3$, $\cdots$, *then*
$A_0 + A_1 x + A_2 x^2 + \cdots \equiv B_0 + B_1 x + B_2 x^2 + \cdots$; (1)
that is, if in the two members of an equality the coefficients of the like powers of x *are identical, the equality is an identity.*

For by hypothesis we have the identities

$$A_0 \equiv B_0,\ A_1 x \equiv B_1 x,\ A_2 x^2 \equiv B_2 x^2, \cdots$$

Adding these identities, we obtain the identity (1).

262. Conversely, *if*

$$A_0 + A_1 x + A_2 x^2 + \cdots \equiv B_0 + B_1 x + B_2 x^2 + \cdots, \quad (1)$$

then $$A_0 \equiv B_0,\ A_1 \equiv B_1, \cdots;$$

that is, in an identity, the coefficients of the like powers of the variable in the two members are identical.

For A_0 and B_0 are respectively the limits of the equal varying members of (1), as $x \doteq 0$; hence

$$A_0 \equiv B_0. \quad (2)$$

Subtracting (2) from (1), and then dividing by x, we obtain

$$A_1 + A_2 x + A_3 x^2 + \cdots \equiv B_1 + B_2 x + B_3 x^2 + \cdots \quad (3)$$

If $x \doteq 0$, from (3) we obtain

$$A_1 \equiv B_1.$$

In like manner we may prove

$$A_2 \equiv B_2,\ A_3 \equiv B_3, \cdots$$

263. Development of Functions by Undetermined Coefficients.

EXAMPLE 1. Develop $\dfrac{1 - x^2}{1 + x - x^2}$.

Assume
$$\frac{1 - x^2}{1 + x - x^2} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots \qquad (1)$$

Clearing (1) of fractions, and for convenience writing the coefficients of the like powers of x in vertical columns, we obtain

$$1 - x^2 = \begin{array}{l|l|l|l|l} A_0 + A_1 & x + A_2 & x^2 + A_3 & x^3 + A_4 & x^4 + \cdots \\ \quad\ + A_0 & \quad + A_1 & \quad + A_2 & \quad + A_3 & \\ & \quad - A_0 & \quad - A_1 & \quad - A_2 & \end{array} \qquad (2)$$

In the first member of (2), the coefficient of any power of x that does not appear is zero. Equating the coefficients of the like powers of x in the two members of (2), we obtain

$$\left.\begin{array}{r} A_0 = 1, A_0 + A_1 = 0,\ A_2 + A_1 - A_0 = -1,\ A_3 + A_2 - A_1 = 0, \\ A_4 + A_3 - A_2 = 0, \cdots,\ A_n + A_{n-1} - A_{n-2} = 0. \end{array}\right\} (3)$$

Solving the system of equations (3), we obtain

$$\left.\begin{array}{r} A_0 = 1,\ A_1 = -1,\ A_2 = 1,\ A_3 = -2,\ A_4 = A_2 - A_3 = 3, \\ \cdots,\ A_n = A_{n-2} - A_{n-1}. \qquad\qquad\qquad \end{array}\right\} (4)$$

Substituting these values of A_0, A_1, $\cdots$, in (1), we have

$$\frac{1 - x^2}{1 + x - x^2} = 1 - x + x^2 - 2x^3 + 3x^4 + \cdots, \qquad (5)$$

which is an identity for such values of x as render the series convergent.

The values of A_0, A_1, A_2, $\cdots$, A_n, given in (4), render (2) an identity; for they satisfy equations (3), and therefore render the coefficients of like powers of x in (2) identical. Now if

(2) is an identity, then (1), or (5), also is an identity for such values of x as render the series convergent (§ 259).

The *law of coefficients* of the series in (5) is

$$A_n = A_{n-2} - A_{n-1}. \tag{6}$$

By (6) the series can be readily extended to any number of terms. Thus,

$$A_5 = A_3 - A_4 = -2 - 3 = -5,\ A_6 = A_4 - A_5 = 3 + 5 = 8, \cdots$$

EXAMPLE 2. Develop $\dfrac{1}{x^2 - x^3 - x^4}$.

Assume

$$\frac{1}{x^2 - x^3 - x^4} = A_0 x^{-2} + A_1 x^{-1} + A_2 + A_3 x + \cdots \tag{1}$$

Clearing (1) of fractions, we have

$$1 = \begin{array}{r|r|r|r|l} A_0 + A_1 & x + A_2 & x^2 + A_3 & x^3 + A_4 & x^4 + \cdots \\ -A_0 & -A_1 & -A_2 & -A_3 & \\ & -A_0 & -A_1 & -A_2 & \end{array} \tag{2}$$

Equating the coefficients of like powers of x in (2), we have

$$\left.\begin{array}{r} A_0 = 1,\ A_1 - A_0 = 0,\ A_2 - A_1 - A_0 = 0,\ A_3 - A_2 - A_1 = 0, \\ A_4 - A_3 - A_2 = 0, \ldots, A_n - A_{n-1} - A_{n-2} = 0. \end{array}\right\} \tag{3}$$

$$\therefore\ A_0 = 1,\ A_1 = 1,\ A_2 = 2,\ A_3 = 3,\ A_4 = 5, \cdots \tag{4}$$

Here the law of coefficients is $A_n = A_{n-1} + A_{n-2}$.

Substituting in (1) the values in (4), we obtain

$$\frac{1}{x^2 - x^3 - x^4} = x^{-2} + x^{-1} + 2 + 3x + 5x^2 + \cdots, \tag{5}$$

which is an identity for such values of x as render the series convergent. [Let the student give the proof.]

NOTE. The form assumed for the series must in each case be such that when the equality is cleared of fractions, no power of x will appear in the first member which is not also found in

the second. For otherwise, the system of equations obtained by equating the coefficients of the like powers of x will be impossible. For example, let us assume

$$\frac{1}{x^2 - x^3 - x^4} = A_0 + A_1 x + A_2 x^2 + \cdots$$

Clearing of fractions, and equating the coefficients of x^0, we obtain the absurdity $1 = 0$, which shows that we have assumed an impossible form for the development. By the laws of exponents in division we know that the first term of the series will contain x^{-2}; hence we assume the form in (1).

EXERCISE 25.

Develop the following functions by the principle of undetermined coefficients, and verify the results by division:

1. $\dfrac{1 + x}{1 - 2x + 3x^2}$.
2. $\dfrac{1 + 2x - 3x^2}{1 - 3x^2}$.
3. $\dfrac{x^4 + 2x^2 - 1}{1 - x + x^2}$.
4. $\dfrac{x^3 + x^2 + 3}{1 + x + x^2}$.
5. $\dfrac{1 + x}{2x^2 + 3x^3}$.
6. $\dfrac{1 + x^2}{x^3 + 3x^4}$.

Resolution of Fractions into Partial Fractions.

264. In elementary Algebra a group of fractions connected by the signs + and − are often united into a single fraction whose denominator is the lowest common denominator of the given fractions (§ 64).

The converse problem of separating a *rational* fraction into a group of simpler, or *partial*, real fractions frequently occurs. The denominators of these partial fractions must evidently be the real factors of the denominator of the given fraction. These real factors may be

I. Linear and unequal.

II. Linear and some of them equal.

III. Quadratic and unequal.

IV. Quadratic and some of them equal.

To present the subject as clearly as possible, we shall consider these cases separately.

265. CASE I. *To a linear factor of the denominator, as* x — a, *there corresponds a partial fraction of the form* $\dfrac{\mathrm{A}}{\mathrm{x}-\mathrm{a}}$.

EXAMPLE. Resolve $\dfrac{2x+3}{x^3+x^2-2x}$ into partial fractions.

Assume
$$\frac{2x+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}. \qquad (1)$$

Clearing (1) of fractions, we have

$$2x+3=A(x-1)(x+2)+B(x+2)x+C(x-1)x \qquad (2)$$
$$=(A+B+C)x^2+(A+2B-C)x-2A. \qquad (3)$$

Equating the coefficients of like powers of x in (3), we have

$$A+B+C=0,\ A+2B-C=2,\ -2A=3. \qquad (4)$$

Solving equations (4), we find

$$A = -\tfrac{3}{2},\ B = \tfrac{5}{3},\ C = -\tfrac{1}{6}. \qquad (5)$$

Substituting these values in (1), we obtain

$$\frac{2x+3}{x^3+x^2-2x} \equiv -\frac{3}{2x} + \frac{5}{3(x-1)} - \frac{1}{6(x+2)}. \qquad (6)$$

The values of A, B, and C, given in (5), render (3) and therefore (1) an identity; for they satisfy (4), and therefore render the coefficients of like powers of x in (3) identical; hence (6) is an identity.

If we assume that (2) is an identity, the values of A, B, and C may be obtained as follows:

Making $x = 0$, (2) becomes $3 = -A$; $\therefore A = -\tfrac{3}{2}$.

Making $x = 1$, (2) becomes $5 = 3B$; $\therefore B = \tfrac{5}{3}$.

Making $x = -2$, (2) becomes $-1 = 6C$; $\therefore C = -\tfrac{1}{6}$.

266. CASE II. *To* r *equal linear factors of the denominator as* $(x - b)^r$, *there corresponds a series of* r *partial fractions of the form*

$$\frac{A}{(x-b)^r} + \frac{B}{(x-b)^{r-1}} + \cdots + \frac{L}{x-b}.$$

EXAMPLE. Resolve $\dfrac{1}{(x-1)^2(x+1)}$ into partial fractions.

Assume

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}. \qquad (1)$$

Clearing (1) of fractions, we have

$$\begin{aligned} 1 &= A(x+1) + B(x-1)(x+1) + C(x-1)^2 \\ &= (B+C)x^2 + (A-2C)x + A - B + C. \qquad (2) \end{aligned}$$

Equating the coefficients of like powers of x, we have

$$B + C = 0,\ A - 2C = 0,\ A - B + C = 1. \qquad (3)$$

Hence, $$A = \tfrac{1}{2},\ B = -\tfrac{1}{4},\ C = \tfrac{1}{4}. \qquad (4)$$

Substituting these values in (1), we have

$$\frac{1}{(x-1)^2(x+1)} = \frac{1}{2(x-1)^2} + \frac{1}{4(1-x)} + \frac{1}{4(x+1)}. \qquad (5)$$

Equality (5) is an identity; for the values of A, B, and C given in (4) satisfy (3), and hence render (2) and therefore (1) an identity.

267. CASE III. *To any quadratic factor of the denominator, as* $x^2 + px + q$, *there corresponds a partial fraction of the form* $\dfrac{Ax + B}{x^2 + px + q}$.

EXAMPLE. Resolve $\dfrac{x^2}{x^4 + x^2 - 2}$ into partial fractions.

Assume

$$\frac{x^2}{(x^2+2)(x+1)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1} + \frac{D}{x-1}. \qquad (1)$$

Clearing (1) of fractions, we obtain

$$\begin{aligned} x^2 &= (Ax+B)(x^2-1) + C(x^2+2)(x-1) + D(x^2+2)(x+1) \\ &= (A+C+D)x^3 + (D-C+B)x^2 + (2C+2D-A)x + 2D-2C-B. \qquad (2) \end{aligned}$$

Equating coefficients of like powers of x in (2), we have

$$\left.\begin{aligned} A + C + D = 0,\ D - C + B = 1, \\ 2C + 2D - A = 0,\ 2D - 2C - B = 0. \end{aligned}\right\} \qquad (3)$$

Whence, $$A = 0,\ B = \tfrac{2}{3},\ C = -\tfrac{1}{6},\ D = \tfrac{1}{6}. \qquad (4)$$

Substituting these values in (1), we have

$$\frac{x^2}{x^4 + 2x - 2} \equiv \frac{2}{3(x^2+2)} - \frac{1}{6(x+1)} + \frac{1}{6(x-1)}. \qquad (5)$$

(5) is an identity for the same reason as that given above.

268. CASE IV. *To* r *equal quadratic factors of the denominator, as* $(x^2 + px + q)^r$, *there corresponds* r *partial fractions of the form*

$$\frac{Ax+B}{(x^2+px+q)^r}+\frac{Cx+D}{(x^2+px+q)^{r-1}}+\cdots+\frac{Lx+M}{x^2+px+q}.$$

In any example under this case, by clearing the assumed equation of fractions and equating the coefficients of like powers of x, we would, as in the first three cases, evidently obtain as many simple equations as there are undetermined quantities, and the values of A, B, C, ..., M, thus determined would make the assumed equality an identity.

269. In what precedes, the numerator is supposed to be of a lower degree than the denominator. If this is not the case, the fraction can be separated by division into an entire part and a fraction whose numerator is of a lower degree than its denominator.

For example:

$$\frac{x^4}{x^3+2x^2-x-2}\equiv x-2+\frac{5x^2-4}{x^3+2x^2-x-2},$$

and $$\frac{5x^2-4}{x^3+2x^2-x-2}\equiv\frac{16}{3(x+2)}-\frac{1}{2(x+1)}+\frac{1}{6(x-1)}.$$

$$\therefore\ \frac{x^4}{x^3+2x^2-x-2}\equiv x-2+\frac{16}{3(x+2)}-\frac{1}{2(x+1)}+\frac{1}{6(x-1)}.$$

EXERCISE 26.

Resolve the following fractions into partial fractions:

1. $\dfrac{x^2-2}{x-x^3}$.

2. $\dfrac{x+1}{x^2-7x+12}$.

3. $\dfrac{x+3}{x^2+x-2}$.

4. $\dfrac{1+3x+2x^2}{(1-2x)(1-x^2)}$.

5. $\dfrac{2x^3-12x^2-8x+2}{x^4-5x^2+4}$.

6. $\dfrac{9}{(x-1)(x+2)^2}$.

7. $\dfrac{3x^2-7x+6}{(x-1)^3}$.

8. $\dfrac{x^2+2x}{(x-2)^2(x+3)^2}$.

9. $\dfrac{23x-11x^2}{(2x-1)(9-x^2)}$.

10. $\dfrac{18x^2+12x-3}{(3x+2)^3}$.

11. $\dfrac{42-19x}{(x^2+1)(x-4)}$.

12. $\dfrac{x^3+x-1}{(x^2+2)^2}$.

Assume $\dfrac{x^3+x-1}{(x^2+2)^2}=\dfrac{Ax+B}{(x^2+2)^2}+\dfrac{Cx+D}{x^2+2}$.

13. $\dfrac{2x^2-11x+5}{(x^2+2x-5)(x-3)}$.

14. $\dfrac{x^2-2x+3}{(x^2+1)^2}$.

* **270. Reversion of Series.** Given

$$y=ax+bx^2+cx^3+\cdots, \qquad (1)$$

the series being convergent, to express x in terms of y; that is, to *revert* the series.

Assume $\quad x=Ay+By^2+Cy^3+\cdots \qquad (2)$

Substituting in (2) the value of y given in (1),

$$x = Aax + \left.\begin{array}{l}Ab\\+Ba^2\end{array}\right| x^2 + \left.\begin{array}{l}Ac\\+2Bab\\+Ca^3\end{array}\right| x^3 + \cdots \quad (3)$$

Equating the coefficients of like powers of x in (3),

$Aa = 1,\ Ab + Ba^2 = 0,\ Ac + 2Bab + Ca^3 = 0, \cdots$

Hence $A = \frac{1}{a},\ B = -\frac{b}{a^3},\ C = \frac{2b^2 - ac}{a^5}, \cdots$

Substituting these values in (2), we obtain

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \cdots,$$

which is the result sought.

If the series to be reverted be of the form

$$y = a_0 + ax + bx^2 + cx^3 + \cdots,$$

we express x in terms of $y - a_0$.

EXAMPLE. Revert the series

$$y = 2 + 2x - x^2 - x^3 + 2x^4 + \cdots \quad (1)$$

From (1), $y - 2 = 2x - x^2 - x^3 + 2x^4 + \cdots \quad (2)$

Assume $x = A(y-2) + B(y-2)^2 + C(y-2)^3 + \cdots \quad (3)$

Substituting in (3) the value of $y - 2$ given in (2),

$$x = 2Ax - \left.\begin{array}{r}A\\+4B\end{array}\right| x^2 - \left.\begin{array}{r}A\\-4B\\+8C\end{array}\right| x^3 + \left.\begin{array}{r}2A\\-3B\\-12C\\+16D\end{array}\right| x^4 + \cdots$$

Equating coefficients of like powers of x, we obtain

$$\left.\begin{array}{l} 2A = 1,\ 4B - A = 0,\ 8C - 4B - A = 0, \\ \qquad 16D - 12C - 3B + 2A = 0, \ldots \end{array}\right\}$$

Hence, $A = \frac{1}{2}$, $B = \frac{1}{8}$, $C = \frac{1}{8}$, $D = \frac{7}{128} \cdots$

Substituting in (3), we have

$$x = \tfrac{1}{2}(y-2) + \tfrac{1}{8}(y-2)^2 + \tfrac{1}{8}(y-2)^3 + \tfrac{7}{128}(y-2)^4 + \cdots$$

EXERCISE 27.

Revert the following series:

1. $y = x + x^2 + x^3 + \cdots$

2. $y = x - 2x^2 + 3x^3 - \cdots$

3. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$

4. $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$

5. $y = x + 3x^2 + 5x^3 + 7x^4 + \cdots$

6. $y = 2x + 3x^3 + 4x^5 + 5x^7 + \cdots$

MACLAURIN'S FORMULA.

271. *Maclaurin's Formula* is a formula for developing a function of a single variable in a series of terms arranged according to the ascending powers of that variable, with constant coefficients.

272. *To deduce Maclaurin's Formula.*

We are to find the values of $A_0, A_1, A_2, \ldots$, when $f(x)$ can be developed in the form

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \cdots \quad (1)$$

in which $A_0, A_1, A_2, \ldots$, are constants, the series being finite, or infinite and convergent.

Differentiating (1) successively and dividing by dx, we obtain

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \cdots \quad (2)$$

$$f''(x) = 2A_2 + 2 \cdot 3 A_3 x + 3 \cdot 4 A_4 x^2 + \cdots \quad (3)$$

$$f'''(x) = 2 \cdot 3 A_3 + 2 \cdot 3 \cdot 4 A_4 x + \cdots \quad (4)$$

$$f^{\text{IV}}(x) = 2 \cdot 3 \cdot 4 A_4 + \cdots \quad (5)$$

Let $x = 0$; then from equations (1) to (5), we obtain

$$f(0) = A_0, \qquad f'(0) = A_1, \qquad f''(0) = 2A_2,$$

$$f'''(0) = \underline{|3} A_3, \quad f^{\text{IV}}(0) = \underline{|4} A_4, \quad \cdots$$

Solving these equations for $A_0, A_1, A_2, \ldots$, we have

$$A_0 = f(0), \qquad A_1 = f'(0), \qquad A_2 = \frac{f''(0)}{\underline{|2}},$$

$$A_3 = \frac{f'''(0)}{\underline{|3}}, \quad A_4 = \frac{f^{\text{IV}}(0)}{\underline{|4}}. \quad \therefore \; A_{n-1} = \frac{f^{n-1}(0)}{\underline{|n-1}}.$$

Substituting these values in (1), we obtain

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + f'''(0)\frac{x^3}{\underline{|3}} + \cdots + f^{n-1}(0)\frac{x^{n-1}}{\underline{|n-1}} + \cdots \quad (6)$$

which is the required formula.

This formula, though bearing the name of Maclaurin, was first discovered by James Stirling in the early part of the last century.

The Binomial Theorem, Logarithmic Series, Exponential Series, and many other formulas, are but particular cases of this more general formula.

BINOMIAL THEOREM.

273. The **Binomial Theorem** is a formula by which a binomial with any exponent may be expanded in a series. Its general demonstration was first given by Sir Isaac Newton. It was considered one of the finest of his discoveries, and was engraved on his tomb.

274. *To deduce the Binomial Theorem.*

To do this we develop $(a + x)^m$ by the formula

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + \cdots + f^{n-1}(0)\frac{x^{n-1}}{\underline{|n-1}} + \cdots \quad (1)$$

Here $f(x) = (a + x)^m$; $\quad\therefore\ f(0) = a^m.$

$f'(x) = m\,(a + x)^{m-1}$; $\quad\therefore\ f'(0) = m\,a^{m-1}.$

$f''(x) = m\,(m - 1)\,(a + x)^{m-2}$;

$\therefore\ f''(0) = m\,(m - 1)\,a^{m-2}.$

$f'''(x) = m\,(m - 1)\,(m - 2)\,(a + x)^{m-3}$;

$\therefore\ f'''(0) = m\,(m - 1)\,(m - 2)\,a^{m-3}.$

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

$f^{n-1}(x) = m\,(m - 1) \cdots (m - n + 2)\,(a + x)^{m-n+1}$;

$\therefore\ f^{n-1}(0) = m\,(m - 1) \cdots (m - n + 2)\,a^{m-n+1}.$

Substituting these values in formula (1), we have

$$(a+x)^m = a^m + m\,a^{m-1}x + \frac{m(m-1)}{\lfloor 2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{\lfloor 3}a^{m-3}x^3$$

$$+ \cdots + \frac{m(m-1)\cdots(m-n+2)}{\lfloor n-1}a^{m-n+1}x^{n-1} + \cdots,$$

in which the last term is the nth, or general, term of the formula.

Example. Find the 6th term in the expansion of $(x^2 - b^{\frac{1}{2}})^{-\frac{2}{3}}$.

Here $n = 6$, $a = x^2$, $x = -b^{\frac{1}{2}}$, and $m = -\frac{2}{3}$; hence $m - n + 2 = -\frac{14}{3}$. Substituting these values in the nth term of the formula, we obtain

$$\text{6th term} = \frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})(-\frac{11}{3})(-\frac{14}{3})}{1\cdot 2\cdot 3\cdot 4\cdot 5}(x^2)^{-\frac{2}{3}-5}(-b^{\frac{1}{2}})^5$$

$$= \frac{308}{729}x^{-\frac{34}{3}}b^{\frac{5}{2}}.$$

275. By an inspection of the Binomial Theorem we discover the following laws of exponents and coefficients, which are very useful in its applications:

(i.) *The exponent of* a *in the first term of the series is the same as that of the binomial, and it decreases by unity in each succeeding term.*

(ii.) *The exponent of* x *is unity in the second term, and increases by unity in each succeeding term.*

(iii.) *The coefficient of the first term is unity, and that of the second is the exponent of the binomial.*

(iv.) *If in any term the coefficient be multiplied by the exponent of* a, *and this product be divided by the exponent of* x *increased by unity, the result will be the coefficient of the next term.*

EXAMPLE 1. Expand $(c+3y)^4$.

Here $\quad a=c,\ x=3y,\ m=4;$

$$\therefore\quad (c+3y)^4 \equiv c^4+4c^3(3y)+6c^2(3y)^2+4c(3y)^3+c^0(3y)^4 \quad (1)$$

$$\equiv c^4+12c^3y+54c^2y^2+108cy^3+81y^4. \quad (2)$$

In the series in (1) the exponent of c in the 5th term is 0; hence by (iv.) the 6th term is 0, and therefore the expansion consists of 5 terms.

EXAMPLE 2. Expand $(n^2-c^2)^{-\frac{1}{3}}$, or $[n^2+(-c^2)]^{-\frac{1}{3}}$.

Applying the laws given above, noting that here $a=n^2$, $x=-c^2$, and $m=-\frac{1}{3}$, we have

$$(n^2-c^2)^{-\frac{1}{3}}=(n^2)^{-\frac{1}{3}}-\tfrac{1}{3}(n^2)^{-\frac{4}{3}}(-c^2)+\tfrac{2}{9}(n^2)^{-\frac{7}{3}}(-c^2)^2-\tfrac{14}{81}(n^2)^{-\frac{10}{3}}(-c^2)^3+\dots \quad (1)$$

$$=n^{-\frac{2}{3}}+\tfrac{1}{3}n^{-\frac{8}{3}}c^2+\tfrac{2}{9}n^{-\frac{14}{3}}c^4+\tfrac{14}{81}n^{-\frac{20}{3}}c^6+\dots \quad (2)$$

By (iv.) the coefficient of the 3d term in (1) is $(-\frac{1}{3})(-\frac{4}{3}) \div 2$, or $\frac{2}{9}$; that of the 4th term is $\frac{2}{9}(-\frac{7}{3}) \div 3$, or $-\frac{14}{81}$, etc.

In (1) the 2d term has two negative factors; the 3d, two; the 4th, four, etc.; hence the signs of all the terms in (2) are $+$.

This development could be obtained by substituting n^2, $-c^2$, and $-\frac{1}{3}$, respectively, for a, x, and m, in the formula, but the process would be longer.

In the series in (1), the exponent of n^2 cannot be 0 in any term; hence no term can have 0 as a factor of its coefficient, and thus vanish. Therefore the expansion is an infinite series, and equals the function only when convergent.

276. *When* m *is a positive whole number, the binomial series is finite and consists of* m + 1 *terms; when* m *is fractional or negative, the series is infinite.*

For when m is a positive whole number, the exponent of a in the $(m+1)$th term is 0; hence by (iv.) of § 275 the $(m+2)$th term and all succeeding terms are 0. Therefore the series consists of $m+1$ terms.

But when m is fractional or negative, the exponent of a cannot be 0 in any term; hence no term can have 0 as a factor, and the series is infinite.

Thus, the expansion of $(x+y)^{13}$ is a finite series of 14 terms; while the expansion of $(x+y)^{\frac{1}{2}}$ or of $(x+y)^{-2}$ is an infinite series.

277. *When* m *is a positive whole number, the coefficients of terms equidistant from the beginning and end of the expansion of* $(a+x)^m$ *are equal.*

For

$$(a+x)^m \equiv a^m + m a^{m-1} x + \frac{m(m-1)}{\lfloor 2} a^{m-2} x^2 + \cdots + \frac{\lfloor m}{\lfloor m} x^m, \quad (1)$$

and

$$(x+a)^m \equiv x^m + m x^{m-1} a + \frac{m(m-1)}{\lfloor 2} x^{m-2} a^2 + \cdots + \frac{\lfloor m}{\lfloor m} a^m. \quad (2)$$

Now the series in (2) has the same terms as the series in (1), but in reverse order; whence the proposition. Hence, in expanding any positive power of a binomial, after we have computed the coefficients of the first half of the series, the remaining coefficients are known to be those already found written in reverse order.

EXERCISE 28.

1. If m is a positive integer, what is the sign of the even terms in $(a - x)^m$? Why?

2. Write out the expansion of $(1 + x)^m$.

Expand

3. $(c + d)^6$.

4. $(3x + 2y)^4$.

5. $(c^2 + b^2)^5$.

6. $(2 - \frac{3}{2}x^2)^4$.

7. $\left(r^{-2} - b^{\frac{1}{2}}\right)^4$.

8. $\left(r^2 - 3n^{-\frac{1}{2}}\right)^5$.

9. $\left(\frac{2}{3}x - \frac{3}{2x}\right)^6$.

10. $(\frac{1}{2} + a)^8$.

11. $\left(1 - \frac{1}{x}\right)^{10}$.

12. $\left(r^{-\frac{1}{3}} - s^{-\frac{1}{2}}\right)^6$.

13. $\left(a^{-\frac{3}{5}} - 2b^2c^{\frac{1}{3}}\right)^4$.

14. $(1 - xy)^7$.

Expand to five terms

15. $(2 + x^3)^{\frac{2}{3}}$.

16. $(8 + 12a)^{\frac{2}{3}}$.

17. $(y + z)^{-3}$.

18. $(1 + x^2)^{-2}$.

19. $(1 - 3x)^{\frac{1}{3}}$.

20. $(1 - 3x)^{-\frac{1}{3}}$.

21. $\left(a^2 + c^{\frac{1}{2}}\right)^{\frac{3}{4}}$.

22. $(c - d^2)^{-\frac{3}{5}}$.

23. $\left(b^{\frac{1}{2}} - c^{-\frac{3}{2}}\right)^{-\frac{3}{4}}$.

24. $(9 + 2x)^{\frac{1}{2}}$.

25. $(4a - 8x)^{-\frac{1}{2}}$.

26. $\left(c^2a^{-\frac{1}{2}} - b^2e^{-\frac{2}{3}}\right)^{-\frac{3}{5}}$.

27. $\frac{1}{\sqrt{x^2 - y^3}}$.

28. $\frac{1}{a^{\frac{2}{3}} - b^{-\frac{1}{2}}}$.

29. $\frac{a}{\left(cb^{-\frac{2}{3}} - x^2y^{-\frac{1}{2}}\right)^{\frac{3}{4}}}$.

Find

30. The 4th term of $(x - 5)^{13}$.

31. The 10th term of $(1 - 2x)^{12}$.

32. The 5th term of $\left(2a - \dfrac{b}{3}\right)^8$.

33. The 7th term of $\left(\dfrac{4x}{5} - \dfrac{5}{2x}\right)^9$.

34. The 6th term of $(b^2 - c^2x^2)^{-\frac{1}{2}}$.

35. The 5th term of $(c^{-2} + e^{-\frac{1}{2}})^{-4}$.

36. The 7th term of $(a^2 - b^{-2})^{-\frac{2}{3}}$.

37. The 6th term of $(x^{-\frac{2}{3}} - a^2b^{\frac{3}{4}})^{-\frac{3}{5}}$.

278. *To find the ratio of the* $(n+1)$*th term to the* n*th.*

Substituting $n + 1$ for n in the nth term of the binomial theorem, we obtain as the $(n + 1)$th term,

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{\underline{|n}}\, a^{m-n}x^n.$$

Dividing this by the nth term, we obtain

$$\left(\frac{m-n+1}{n}\right)\frac{x}{a}, \text{ or } \left(\frac{m+1}{n} - 1\right)\frac{x}{a}, \qquad (1)$$

as the ratio sought; that is, (1) is the quantity by which we multiply the nth term to obtain the next term.

This ratio affords the following simple proof of the principle in § 276:

When m is a positive integer, this ratio is evidently zero, for $n = m + 1$; hence the $(m + 2)$th term and all the succeeding terms are zero, and therefore the series consists of $m + 1$ terms.

But when m is fractional or negative, no value of n (n must be integral) will make the ratio zero; hence no term can become zero, and the series is infinite.

279. Any root of a number may be found approximately by the Binomial Theorem.

EXAMPLE. Find the approximate 5th root of 248.

$$\begin{aligned}\sqrt[5]{248} &= (243 + 5)^{\frac{1}{5}}\\ &= (3^5 + 5)^{\frac{1}{5}}\\ &= 3\left(1 + \frac{5}{3^5}\right)^{\frac{1}{5}}\\ &= 3\left(1 + \frac{1}{3^5} - \frac{2}{3^{10}} + \frac{2 \cdot 3}{3^{15}} - \cdots\right)\\ &= 3\,(1 + 0.0041152 - 0.0000338 + 0.0000004 - \cdots)\\ &= 3.0122454,\end{aligned}$$

which is correct to at least six places of decimals.

280. Expressions which contain more than two terms may be expanded by the Binomial Theorem.

EXAMPLE. Find the expansion of $(x^2 + 2x - 1)^3$.

Regarding $2x - 1$ as a single term, we have

$$\begin{aligned}[x^2+(2x-1)]^3 &= (x^2)^3+3(x^2)^2(2x-1)+3x^2(2x-1)^2+(2x-1)^3\\ &= x^6+6x^5+9x^4-4x^3-9x^2+6x-1.\end{aligned}$$

EXERCISE 29.

Expand and write the nth term of

1. $(1-x)^{-1}$.   2. $(1-x)^{-2}$.   3. $(1-x)^{-3}$.

Find to five places of decimals the value of

4. $\sqrt[3]{31}$.   6. $\sqrt[5]{29}$.   8. $\sqrt[4]{2400}$.

5. $\sqrt[4]{17}$.   7. $\sqrt[3]{998}$.   9. $\sqrt[5]{3128}$.

Find the expansion of

10. $(1+2x-x^2)^4$.   11. $(3x^2-2ax+3a^2)^3$.

Find the nth term of the expansion of

12. $\dfrac{1}{3(1-2x)}$.   14. $\dfrac{4}{(2-x)^2}$.

13. $\dfrac{5}{3(2-x)}$.   15. $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$.

By § 266, $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}=-\dfrac{1}{3(1-2x)}+\dfrac{5}{3(2-x)}-\dfrac{4}{(2-x)^2}$.

Hence, by Examples 12, 13, and 14, the nth term is

$$\left(-\frac{2^{n-1}}{3}+\frac{5}{3}\cdot\frac{1}{2^n}-\frac{n}{2^{n-1}}\right)x^{n-1}.$$

16. $\dfrac{1+3x}{1+11x+28x^2}$.   17. $\dfrac{2x-4}{(1-x^2)(1-2x)}$.

Expand to four terms in ascending powers of x

18. $\dfrac{5x+6}{(2+x)(1-x)}$.   19. $\dfrac{2x+1}{(x-1)(x^2+1)}$.

CHAPTER XV.

CONVERGENCY AND SUMMATION OF SERIES.

281. An infinite series is divergent, if the nth term does not approach zero as its limit when $n = \infty$. For if the nth term does not approach zero when $n = \infty$, the sum of n terms cannot approach a limit.

Thus, the series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots \pm \frac{n+1}{n} \mp \cdots$ is divergent; for the nth term approaches unity and not zero as its limit.

A series may be divergent even though the nth term approaches zero as its limit when $n = \infty$.

EXAMPLE. Show that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \text{ is divergent.}$$

If after the first two, the terms of this series be taken in groups of two, four, eight, sixteen, etc., we have

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \quad (1)$$

Each parenthetical expression is evidently greater than $\frac{1}{2}$. Regarding these as single terms of series (1), the sum of m terms is greater than $\frac{1}{2}m$. But m increases indefinitely with n. Hence the series is divergent, although its nth term $\doteq 0$, when $n = \infty$.

282. The following three important principles are almost self-evident:

(i.) An infinite series of positive terms is convergent, if the sum of its first n terms is always less than some finite quantity, however large n may be.

For as the sum of n terms must always increase with n, but cannot exceed a finite value, it must approach some finite limit.

(ii.) If a series in which all the terms are positive is convergent, then the series is convergent when some or all of the terms are negative.

(iii.) If, after removing a finite number of its terms, a series is convergent, the entire series is convergent; if divergent, the entire series is divergent. For the sum of this finite number of terms is finite.

283. *An infinite series in which the terms are alternately positive and negative is convergent, if its terms decrease numerically, and the limit of its* n*th term is zero.*

Let the terms of the series be denoted by u_1, $-u_2$, u_3, . . ., and their sum by s; then

$$s \doteq u_1 - u_2 + u_3 - u_4 + u_5 - \cdots \pm u_n \mp \cdots \quad (1)$$

Since $u_n \doteq 0$ when $n = \infty$, the sum of the series is evidently the same whether we take an even or an odd number of terms.

Now (1) may be written in the form

$$s = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots \tag{2}$$

or $$s = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots \tag{3}$$

Since $u_1 > u_2 > u_3 > \cdots > u_n$, the expressions $u_1 - u_2,\ u_2 - u_3,\ u_3 - u_4, \cdots$ are all positive. Hence from (2) we know that $s < u_1$; therefore by (i.) of § 282 the series in (3) is convergent.

Thus, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \cdots \text{ is convergent.}$$

If we put this series in the forms

$$1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots, \text{ and } \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

we see that its sum is less than 1 and greater than $\frac{1}{2}$.

284. *An infinite series is convergent if the ratio of each term to the preceding term is less than some fixed quantity that is itself numerically less than unity.*

Let all the terms be positive; then

$$s = u_1 + u_2 + u_3 + u_4 + \cdots$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_2}{u_1}\cdot\frac{u_3}{u_2} + \frac{u_2}{u_1}\cdot\frac{u_3}{u_2}\cdot\frac{u_4}{u_3} + \cdots\right). \tag{1}$$

Let k be a fixed quantity less than 1, but greater than any of the ratios $\frac{u_2}{u_1}, \frac{u_3}{u_2}, \frac{u_4}{u_3}, \ldots$; then from (1)

$$s < u_1(1 + k + k^2 + k^3 + \cdots),$$

or $$s < u_1 \frac{1}{1-k}, \text{ a finite quantity.} \qquad \S\ 259, \text{ Ex. } 2.$$

Hence by (i.) and (ii.) of § 282 the series is convergent whether its terms are all positive or some or all negative.

285. *An infinite series is divergent if the ratio of each term to the preceding term is numerically equal to or greater than unity.*

For if this ratio is unity, or greater than unity, the nth term cannot approach zero as its limit, and the series is divergent by § 281.

286. In the application of the tests of §§ 284, 285, it is convenient to find $\underset{n=\infty}{\text{limit}}\left[\frac{u_{n+1}}{u_n}\right]$; let this limit be denoted by r.

If $r < 1$, the series is convergent. § 284.

If $r > 1$, the series is divergent. § 285.

If $r = 1$, and $u_{n+1} \div u_n > 1$, the series is divergent by § 285; if $r = 1$, and $u_{n+1} \div u_n < 1$, the test of § 284 fails, and other tests must be applied.

EXAMPLE 1. For what values of x is the logarithmic series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n} \mp \frac{x^{n+1}}{n+1} \cdots \qquad (1)$$

convergent?

Here $\underset{n=\infty}{\text{limit}}\left[\frac{u_{n+1}}{u_n}\right] = \underset{n=\infty}{\text{limit}}\left[\left(\frac{1}{n+1} - 1\right)x\right] = -x.$

Hence, if $x < 1$ numerically, the series is convergent.

If $x > 1$ numerically, the series is divergent.

If $x = 1$, the series is convergent by § 283.

If $x = -1$, the series becomes $-(1 + \frac{1}{2} + \frac{1}{3} + \cdots)$, and is divergent by Example of § 281.

Hence (1) is convergent for $x = 1$, or $x > -1$ and $< +1$.

EXAMPLE 2. When the binomial series is infinite, for what values of x is it convergent?

Here $\underset{n=\infty}{\text{limit}}\left[\frac{u_{n+1}}{u_n}\right] = \underset{n=\infty}{\text{limit}}\left[\left(\frac{m+1}{n} - 1\right)\frac{x}{a}\right] = -\frac{x}{a}.$ § 278.

Hence, if $x < a$ numerically, the series is convergent.

If $x > a$ numerically, the series is divergent.

If $x = a$ numerically, the test of § 284 fails.

Thus, the theorem will develop $(8 + 2)^{\frac{1}{2}}$, but not $(2 + 8)^{\frac{1}{2}}$.

Hence when m is fractional or negative, the binomial theorem will give the development of $(a + x)^m$ or $(x + a)^m$, according as $x <$ or $> a$ numerically. If $x = a$ numerically, $(a+x)^m$ becomes $(2a)^m$ or 0^m, and the formula is not needed.

EXAMPLE 3. For what values of x is the series

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots + \frac{1}{n^x} + \cdots \text{ convergent?}$$

(i.) If $x > 1$, the first term is 1; the sum of the next two terms is less than $\frac{2}{2^x}$; the sum of the next four terms is less than $\frac{4}{4^x}$; the sum of the next eight terms is less than $\frac{8}{8^x}$; and so on. Hence the sum of the series is less than that of $1 + \frac{2}{2^x} + \frac{4}{4^x} + \frac{8}{8^x} + \cdots$, which is a geometrical progression whose common ratio, $2 \div 2^x$, is less than 1; hence the series is convergent.

(ii.) If $x = 1$, the series is the harmonic series, and is divergent by Example of § 281.

(iii.) If $x < 1$, each term is greater than in case (ii.), and therefore the series is divergent.

EXERCISE 30.

Determine which of the following series is convergent and which divergent:

1. $1 + \frac{2^2}{\lfloor 2} + \frac{3^2}{\lfloor 3} + \frac{4^2}{\lfloor 4} + \cdots$

2. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \cdots$

3. $1 + 2x + 3x^2 + 4x^3 + \cdots$

4. $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$

5. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

6. $1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots$

7. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$

8. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \cdots$

9. $\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \frac{x^4}{7 \cdot 8} + \cdots$

10. $2x + \frac{3x^2}{2^3} + \frac{4x^3}{3^3} + \cdots + \frac{(n+1)x^n}{n^3} + \cdots$

SUMMATION OF SERIES.

287. The **Summation** of a series is the process of finding an expression for the sum of all its terms.

The summation of arithmetical and geometrical series has been given in Chapter XI. We proceed to consider methods of summing other series.

RECURRING SERIES.

288. **A Recurring Series** is one in which each term after the mth term bears a uniform relation to the m preceding terms. If $m = 1$, the series is of the *first order;* if $m = 2$, it is of the *second order*, and so on.

A geometrical progression is a recurring series of the first order; for each term after the first is equal to the preceding term multiplied by the ratio of the series.

The series $1 + 2x + 8x^2 + 28x^3 + 100x^4 + \cdots$ (1)

is a recurring series of the *second order;* for each term after the 2d is the sum of the two preceding terms multiplied respectively by $3x$ and $2x^2$.

Thus, $28x^3 = 3x \cdot 8x^2 + 2x^2 \cdot 2x;$

that is, $u_4 = 3xu_3 + 2x^2u_2,$

and generally after the 2d term, the nth term is connected with the two immediately preceding it by the relation

$$u_n = 3xu_{n-1} + 2x^2u_{n-2},$$

or $$u_n - 3xu_{n-1} - 2x^2u_{n-2} = 0. \quad (2)$$

289. The sum of the coefficients of u_n, u_{n-1}, and u_{n-2} in (2) of § 288 forms what is called the **Scale of Relation** of series (1). Thus the scale of relation of series (1) is $1 - 3x - 2x^2$.

If we have given the scale of relation of a recurring series of the mth order, any term can be found if we know the m preceding terms.

EXAMPLE. Find the 6th term of series (1) in § 288.

$$u_6 = 3x \cdot 100x^4 + 2x^2 \cdot 28x^3 = 356x^5.$$

290. *To find the scale of relation of a recurring series.*

(i.) If the series is of the first order, let $1 - p$ be the scale of relation; then

$$u_2 - pu_1 = 0, \text{ or } p = u_2 \div u_1.$$

(ii.) If the series is of the second order, let $1 - p - q$ be the scale of relation; then

$$u_3 - p\,u_2 - q\,u_1 = 0,$$

and $$u_4 - p\,u_3 - q\,u_2 = 0.$$

From these two equations the values of p and q may be found when the first four terms of the series are known.

(iii.) If the series is of the third order, let $1 - p - q - r$ be the scale of relation; then from any six consecutive terms we can obtain three equations which will determine the values of p, q, and r.

If the series is of the mth order, we must have given $2m$ consecutive terms to find the scale of relation.

EXAMPLE. Find the scale of relation of the recurring series

$$2 + 5x + 13x^2 + 35x^3 + \cdots$$

Let the scale of relation be $1 - p - q$; then to obtain p and q we have the equations,

$$13x^2 - 5px - 2q = 0, \text{ and } 35x^3 - 13px^2 - 5qx = 0.$$

Hence $p = 5x$, and $q = -6x^2$; therefore the scale of relation is $1 - 5x + 6x^2$.

REMARK. In finding the scale of relation, if we assume too high an order for the series we shall find one or more of the assumed multipliers to be zero. If too low an order is assumed, the error will be discovered in attempting to apply the scale when found.

291. *To find the sum of a recurring series.*

If the series is of the first order it is a geometrical progression, and its sum is given in § 211.

If the series is of the second order, we have

$$\left.\begin{aligned} u_1 &= u_1, \\ u_2 &= u_2, \\ u_3 &= p u_2 + q u_1, \\ u_4 &= p u_3 + q u_2, \\ \dots & \quad \dots \\ u_n &= p u_{n-1} + q u_{n-2}. \end{aligned}\right\} \qquad (1)$$

Adding these equalities and denoting the sum of n terms by S_n, we have

$$S_n = u_1 + u_2 + p(S_n - u_1 - u_n) + q(S_n - u_{n-1} - u_n).$$

$$\therefore\ S_n = \frac{u_1(1-p) + u_2}{1-p-q} - \frac{p u_n + q(u_{n-1} + u_n)}{1-p-q}, \qquad (2)$$

in which $1 - p - q$ is the scale of relation.

If the series is infinite and convergent, (2) becomes

$$S_\infty = \frac{u_1(1-p) + u_2}{1-p-q}. \qquad (3)$$

If S_∞ alone had been desired, u_n and u_{n-1} could have been neglected in adding equalities (1).

Similarly, if the series is of the third order, we obtain

$$S_\infty = \frac{u_1(1-p-q) + u_2(1-p) + u_3}{1-p-q-r}, \qquad (4)$$

in which $1 - p - q - r$ is the scale of relation.

292. From (2) and (3) of § 291 we learn that the sum of a recurring series is a fraction whose denominator is the scale of relation of the series.

By developing the fraction in (3) we could obtain as many terms of the original series as we please; for this reason this fraction is called the **Generating Function** of the series.

EXERCISE 31.

Find the generating function of each of the following series:

1. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

   The scale of relation is $1 - 2x + x^2$; $S_\infty = \dfrac{1}{(1-x)^2}$.

2. $1 + 2x + 8x^2 + 28x^3 + 100x^4 + \cdots$

3. $1 + x + 5x^2 + 13x^3 + 41x^4 + \cdots$

4. $1 + 5x + 9x^2 + 13x^3 + \cdots$

5. $2 + 3x + 5x^2 + 9x^3 + \cdots$

6. $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \cdots$

7. $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5 + \cdots$

8. $2 - 2x + 2x^2 - 5x^3 + 10x^4 - 17x^5 + \cdots$

9. $3 + 6x + 14x^2 + 36x^3 + 98x^4 + 276x^5 + \cdots$

METHOD OF DIFFERENCES.

293. If each term of a series be subtracted from its succeeding term, the remainders thus obtained form the *series of first differences;* the remainders obtained by subtracting each term of this series from its succeeding term form the *series of second differences;* and so on.

In an arithmetical series, the second differences vanish. In certain other series, the third, the fourth, the fifth, or the *r*th differences vanish.

Thus, if the series is	1, 4, 9, 16, 25, 36, ...
1st differences,	3, 5, 7, 9, 11, ...
2d differences,	2, 2, 2, 2, ...
3d differences,	0, 0, 0, ...

Here the 3d and all succeeding differences vanish.

294. *To find the* n*th term of the series*

$$u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6, \cdots$$

Here the series of successive differences are

1st, $u_2 - u_1,\ u_3 - u_2,\ u_4 - u_3,\ u_5 - u_4, \cdots$

2d, $u_3 - 2u_2 + u_1,\ u_4 - 2u_3 + u_2,\ u_5 - 2u_4 + u_3, \cdots$

3d, $u_4 - 3u_3 + 3u_2 - u_1,\ u_5 - 3u_4 + 3u_3 - u_2, \cdots$

4th, $u_5 - 4u_4 + 6u_3 - 4u_2 + u_1, \cdots$

... ...

Let D_1, D_2, D_3, ... D_r, denote respectively the first terms of the successive series of differences; then

$$D_1 = u_2 - u_1; \qquad \therefore u_2 = u_1 + D_1.$$

$$D_2 = u_3 - 2u_2 + u_1; \qquad \therefore u_3 = u_1 + 2D_1 + D_2.$$

$$D_3 = u_4 - 3u_3 + 3u_2 - u_1; \therefore u_4 = u_1 + 3D_1 + 3D_2 + D_3.$$

$$D_4 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1;$$

$$\therefore u_5 = u_1 + 4D_1 + 6D_2 + 4D_3 + D_4.$$

... ...

The reader will notice that the coefficients in the value of u_5 are those in the expansion of $(a + x)^4$; a similar relation evidently holds between u_6 and $(a + x)^5$, u_7 and $(a + x)^6$, etc.; hence

$$u_n = u_1 + (n-1)D_1 + \frac{(n-1)(n-2)}{\underline{|2}}D_2 + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}D_3 + \cdots \quad (1)$$

EXAMPLE. Find the 10th term of the series

1, 2, 6, 15, 31, 56, ...

Here the successive series of differences are:

1st differences, 1, 4, 9, 16, 25, ...

2d differences, 3, 5, 7, 9, ...

3d differences, 2, 2, 2, ...

4th differences, 0, 0, ...

Hence $u_1 = 1$, $n = 10$, $D_1 = 1$, $D_2 = 3$, $D_3 = 2$, $D_4 = 0$.

Substituting these values in formula (1), we obtain

$$u_{10} = 1 + 9 + 108 + 168 = 286.$$

295. *To find the sum of* n *terms of the series*

$$u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6, \cdots \qquad (1)$$

Assume the new series

$$0,\ u_1,\ u_1 + u_2,\ u_1 + u_2 + u_3,\ u_1 + u_2 + u_3 + u_4, \cdots \quad (2)$$

Now the sum of n terms of series (1) is evidently equal to the $(n + 1)$th term of series (2). Moreover, the series of first differences of series (2) is series (1); hence the second differences of series (2) are the first differences of series (1); the third differences of series (2) are the second differences of series (1); and so on.

Hence we may obtain the $(n + 1)$th term of series (2), or S_n of series (1), by putting in (1) of § 294

$$u_1 = 0,\ n = n + 1,\ D_1 = u_1,\ D_2 = D_1,\ D_3 = D_2, \cdots$$

Making these substitutions, we have

$$S_n = n u_1 + \frac{n(n-1)}{\lfloor 2} D_1 + \frac{n(n-1)(n-2)}{\lfloor 3} D_2 + \cdots$$

EXAMPLE. Find the sum of n terms of the series

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2, \cdots,\ n^2.$$

1st differences, 3, 5, 7, 9, ⋯

2d differences, 2, 2, 2, ⋯

3d differences, 0, 0, ⋯

Hence $u_1 = 1,\ D_1 = 3,\ D_2 = 2,\ D_3 = 0.$

Substituting these values in the formula, we obtain

$$S_n = n + \frac{n(n-1)}{\lfloor 2} \times 3 + \frac{n(n-1)(n-2)}{\lfloor 3} \times 2$$
$$= \tfrac{1}{6} n (2n^2 + 3n + 1) = \tfrac{1}{6} n (n + 1)(2n + 1).$$

EXERCISE 32.

1. Find the 7th term of the series 3, 5, 8, 12, 17, ...

Ans. 30.

2. Find the 15th term of the series 3, 7, 14, 25, 41, ...

3. Find the 7th term of the series 286, 205, 141, 92, 56, ...

4. Find the 9th term of the series 194, 191, 174, 146, 110, ...

5. Find the nth term of the series 1, 3, 6, 10, 15, 21, ...

Find the sum of each of the following series:

6. $1, 3, 5, 7, 9, \ldots, 2n-1$.

7. $2, 4, 6, 8, \ldots, 2n$.

8. $1^2, 3^2, 5^2, 7^2, \ldots, (2n-1)^2$.

9. $2^2, 4^2, 6^2, 8^2, \ldots, (2n)^2$.

10. $m+1,\ 2(m+2),\ 3(m+3),\ \ldots,\ n(m+n)$.

Ans. $S_n = \frac{1}{6}n(n+1)(3m+2n+1)$.

11. Find the number of balls that can be placed in an equilateral triangle with n on a side; that is, find the sum of the series $1, 2, 3, 4, 5, \ldots, n$. Ans. $\frac{1}{2}n(n+1)$.

12. Obtain the series whose nth term is $\frac{1}{2}n(n+1)$, and find the sum of n terms. Ans. $\frac{1}{6}n(n+1)(n+2)$.

13. Show that

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = (1+2+3+4+\cdots+n)^2.$$

296. Application to Piles of Balls. An interesting application of the preceding theory is that of finding the number of cannon-balls in the triangular and square pyramids, and rectangular piles, in which they are placed in arsenals and navy-yards.

Triangular Piles. When the pile is in the form of a regular triangular pyramid, the top course contains one ball, the second course contains three balls, and the nth course from the top is a triangle of balls with n on a side, and therefore contains $\frac{1}{2}n(n+1)$ balls (Example 11 of Exercise 32). Hence the whole number of balls in a triangular pyramid having n balls on a side of its bottom course is the sum of the series

$$1, 3, 6, 10, 15, 21, \ldots, \tfrac{1}{2}n(n+1).$$

Hence by Example 12 of Exercise 32,

$$S_n = \tfrac{1}{6}n(n+1)(n+2). \qquad (1)$$

Square Piles. When the base of the pile is a square having n balls on a side, the top course contains one ball, the second course 2^2 balls, the third course 3^2 balls, and the nth course n^2 balls. Hence the number of balls in the pile is the sum of the series

$$1^2, 2^2\ 3^2, 4^2, \ldots, n^2.$$

Hence by Example of § 295

$$S_n = \tfrac{1}{6}n(n+1)(2n+1). \qquad (2)$$

Rectangular Piles. When the base of the pile is a rectangle having n balls on one side and $m + n$ on the other, the top course will be a single row of $m + 1$ balls; the second course will contain $2(m + 2)$ balls; the third course $3(m + 3)$ balls; and the bottom course $n(m + n)$ balls.

Hence the number of balls in the pile is the sum of the series

$$m + 1,\ 2(m + 2),\ 3(m + 3),\ \cdots,\ n(m + n).$$

Hence by Example 10 of Exercise 32,

$$S_n = \tfrac{1}{6}n(n+1)(3m + 2n + 1). \qquad (3)$$

If we put $m = 0$, (3) becomes identical with (2), as it should; for when $m = 0$, the pile is a square pyramid.

Incomplete Piles. If the pile is *incomplete*, find the number of balls in the pile supposed complete, then find the number in the part that is lacking, and subtract the last number from the first.

EXERCISE 33.

1. Find the number of balls in a triangular pile of 12 courses. How many balls in the lowest course? How many in one of the faces?

2. If from a triangular pile of 20 courses, 8 courses be removed from the top, how many balls will be left?

3. If from a triangular pile of b courses, c courses be removed from the top, how many balls will be left?

4. How many balls in a square pile of 25 courses? How many balls in each face?

5. How many balls in a square pile having 256 balls in its lowest course?

6. Find the number of balls in the lower 12 courses of a square pile having 20 balls on each side of its lowest course.

7. The top course of an incomplete triangular pile contains 21 balls, and the lowest course has 20 balls on a side. How many balls in the pile?

8. Find the number of balls in an oblong pile whose lowest course is 52 balls in length and 21 in breadth. If 11 courses were removed from the top of this pile, how many balls would be left?

9. Find the number of balls in an incomplete oblong pile whose top course is 10 balls by 30, and whose bottom course is 45 balls in length.

10. Find the number of balls in a rectangular pile which has 11 balls in the top row and 875 in the bottom course.

297. Interpolation is the process of introducing between the terms of a series intermediate terms which conform to the law of the series. It is used in finding terms intermediate between those given in mathematical tables, but its most extensive application is in Astronomy.

The formula for interpolation is that for finding the nth term of the series by the *method of differences.* Thus to find the term equidistant from the 1st and 2d terms of a series we put $n = 1\frac{1}{2}$ in (1) of § 294; to find the term equidistant from the 2d and 3d terms we put $n = 2\frac{1}{2}$.

EXAMPLE 1. Given $\log 97 = 1.9868$, $\log 98 = 1.9912$, $\log 99 = 1.9956$; find $\log 97.32$.

Series,	1.9868,		1.9912,		1.9956.
1st differences,		0.0044,		0.0044.	
2d differences,			0.		

Hence $u_1 = 1.9868$, $D_1 = 0.0044$, $D_2 = 0$, $n = 1.32$.

$$\therefore \quad \log 97.32 = 1.9868 + 0.32 \times 0.0044$$
$$= 1.9882.$$

EXAMPLE 2. Given $\sqrt[3]{45} = 3.55689$, $\sqrt[3]{47} = 3.60882$, $\sqrt[3]{49} = 3.65930$, $\sqrt[3]{51} = 3.70843$; find $\sqrt[3]{48}$.

Here $u_1 = 3.55689$, $D_1 = 0.05193$, $D_2 = -0.00145$, $D_3 = 0.0001$, $n - 1 = \frac{3}{2}$.

Hence

$$\sqrt[3]{48} = 3.55689 + \tfrac{3}{2}(0.05193) + \tfrac{3}{8}(-0.00145) - \tfrac{1}{16}(0.0001)$$
$$= 3.63424.$$

EXERCISE 34.

1. Given $\sqrt{5} = 2.23607$, $\sqrt{6} = 2.44949$, $\sqrt{7} = 2.64575$, $\sqrt{8} = 2.82843$; find $\sqrt{5.08}$, $\sqrt{6.5}$.

2. Given the length of a degree of longitude in latitude $41^\circ = 45.28$ miles; in latitude $42^\circ = 44.59$ miles; in lati-

tude $43^\circ = 43.88$ miles; in latitude $44^\circ = 43.16$ miles. Find the length of a degree of longitude in latitude $42^\circ\ 30'$.

Ans. 44.24 miles.

3. If the amount of \$1 at 7 per cent compound interest for 2 years is \$1.145, for 3 years \$1.225, for 4 years \$1.311, and for 5 years \$1.403, what is the amount for 4 years and 9 months? for 3 years and 6 months?

298. The summation of some series is readily effected by writing the series as the difference of two other series.

EXAMPLE 1. Sum the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)}.$$

Now $\dfrac{1}{1\cdot 2}=1-\dfrac{1}{2}$; $\quad \dfrac{1}{2\cdot 3}=\dfrac{1}{2}-\dfrac{1}{3}$; ...

Writing the positive and negative terms separately, and denoting the sum of n terms of the given series by S_n, we have

$$S_n=\left\{\begin{array}{l} 1+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right) \\ \quad -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)-\frac{1}{n+1} \end{array}\right\}$$

$$=1-\frac{1}{n+1}=\frac{n}{n+1} \qquad (1)$$

If the series is infinite, (1) becomes $S_\infty = 1$.

EXAMPLE 2. Sum the series $\dfrac{1}{1\cdot 4}+\dfrac{1}{2\cdot 5}+\dfrac{1}{3\cdot 6}+\cdots$

Here the nth term is evidently $\dfrac{1}{n(n+3)}$.

Now $\dfrac{1}{n(n+3)}=\dfrac{1}{3}\left(\dfrac{1}{n}-\dfrac{1}{n+3}\right)$; § 265.

$$\therefore\quad \frac{1}{1\cdot 4}=\frac{1}{3}\left(1-\frac{1}{4}\right),\quad \frac{1}{2\cdot 5}=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right),\cdots$$

$$\therefore\quad S_n=\frac{1}{3}\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\left(\frac{1}{4}+\cdots+\frac{1}{n}\right)\\ \qquad\qquad -\left(\frac{1}{4}+\cdots+\frac{1}{n}\right)-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\end{array}\right\}$$

$$=\frac{1}{3}\left(\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right).$$

Hence $S_\infty=\frac{11}{18}$.

EXAMPLE 3. Sum the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots$

Multiplying and dividing by 2, we have

$$S_n=2\left(\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 6}+\frac{1}{2\cdot 10}+\cdots\right),$$

$$=2\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)}\right).$$

$$=\frac{2n}{n+1};\quad \therefore S_\infty=2.$$ Example 1.

EXERCISE 35.

Find the nth term, the sum of n terms, and the sum of all the terms in each of the following series:

1. $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots$

2. $\dfrac{4}{1\cdot 5}+\dfrac{4}{5\cdot 9}+\dfrac{4}{9\cdot 13}+\dfrac{4}{13\cdot 17}+\cdots$

3. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\cdots$

4. $\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$

5. $\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots$

6. $\frac{2}{3 \cdot 8} + \frac{2}{6 \cdot 12} + \frac{2}{9 \cdot 16}$.

7. The series of which the nth term is $\frac{1}{(3n+2)(3n+8)}$.

8. $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots$

9. Sum n terms of the series $1, 2^4, 3^4, 4^4, \cdots$

10. Show that the number of balls in a square pile is one-fourth the number of balls in a triangular pile of double the number of courses.

11. If the number of balls in a triangular pile is to the number of balls in a square pile of double the number of courses as 13 to 175, find the number of balls in each pile.

12. The number of balls in a triangular pile is greater by 150 than half the number of balls in a square pile, the number of courses in each being the same. Find the number of balls in the lowest course of the triangular pile.

13. If from a complete square pile of n courses a triangular pile of the same number of courses be formed, show that the remaining balls will be just sufficient to form another triangular pile, and find the number of its courses.

CHAPTER XVI.

LOGARITHMS.

299. The **Logarithm** of a number is the exponent by which a fixed number, called the *base*, must be affected in order to equal the given number. That is, if $a^x = N$, x is the logarithm of N to the base a, which is written $x = \log_a N$.

Thus, since $3^2 = 9$, $2 = \log_3 9$.

Since $2^4 = 16$, $4 = \log_2 16$.

Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, ...,

the positive numbers 1, 2, 3, ..., are respectively the logarithms of 10, 100, 1000, ..., to the base 10.

To the base 10 the logarithms of all numbers between 1 and 10, 10 and 100, 100 and 1000, ..., are incommensurable.

Since $3^{-2} = \frac{1}{9}$, $-2 = \log_3 \frac{1}{9}$.

Since $10^{-1} = 0.1$, $10^{-2} = 0.01$, $10^{-3} = 0.001$, ...,

the negative numbers -1, -2, -3, ..., are respectively the logarithms of 0.1, 0.01, 0.001, ..., to the base 10.

300. Any positive number except 1 may evidently be taken as the base of logarithms.

The logarithms of all *positive* numbers to any given base constitute a **System of Logarithms**.

In any system, the logarithms of most numbers are incommensurable.

Before discussing the two systems commonly used, we shall prove some general propositions that are true for any system.

301. *The logarithm of* 1 *is* 0.

For $a^0 = 1; \quad \therefore \log_a 1 = 0.$

302. *The logarithm of the base itself is* 1.

For $a^1 = a; \quad \therefore \log_a a = 1.$

303. *The logarithm of a product equals the sum of the logarithms of its factors.*

Let $\log_a M = x, \quad \log_a N = y;$

then $M = a^x, \qquad N = a^y.$ § 299.

Therefore $MN = a^{x+y}.$

Hence $\log_a (MN) = x + y = \log_a M + \log_a N.$

Similarly, $\log_a(MNQ) = \log_a M + \log_a N + \log_a Q;$

and so on, for any number of factors.

304. *The logarithm of a quotient equals the logarithm of the dividend minus that of the divisor.*

Let $M = a^x, \quad N = a^y;$

then $M \div N = a^{x-y}.$

Hence $\log_a (M \div N) = x - y = \log_a M - \log_a N.$

305. *The logarithm of a positive number affected with any exponent equals the logarithm of the number multiplied by the exponent.*

Let $$M = a^x;$$

then, whatever be the value of p,

$$M^p = a^{px}.$$

Hence $$\log_a (M^p) = p\,x = p \log_a M.$$

306. By § 305, the logarithm of any power of a number equals the logarithm of the number multiplied by the exponent of the power; and the logarithm of any root of a number equals the logarithm of the number divided by the index of the root.

307. From the principles proved above, we see that by the use of logarithms the operations of multiplication and division may be replaced by those of addition and subtraction, and the operations of involution and evolution by those of multiplication and division.

Example. Express $\log_a \dfrac{\sqrt{b^3}}{z^2 x^{\frac{2}{3}}}$ in terms of $\log_a b$, $\log_a z$, $\log_a x$.

$$\text{Log}_a \frac{\sqrt{b^3}}{z^2 x^{\frac{2}{3}}} = \log_a b^{\frac{3}{2}} - \log_a (z^2 x^{\frac{2}{3}}) \qquad \text{§ 304.}$$

$$= \log_a b^{\frac{3}{2}} - (\log_a z^2 + \log_a x^{\frac{2}{3}}) \qquad \text{§ 303.}$$

$$= \tfrac{3}{2} \log_a b - 2 \log_a z - \tfrac{2}{3} \log_a x. \qquad \text{§ 305.}$$

308. If $a > 1$, and $a^x = N$;
then if $N > 1$, x is positive;
if $N < 1$, x is negative;
if $N = \infty$, $x = \infty$;
if $N = 0$, $x = -\infty$.

That is, *if the base is greater than unity,*

(i.) *The logarithm is positive or negative according as the number is greater or less than unity.*

(ii.) *The logarithm of an infinite is infinite; and the logarithm of an infinitesimal is a negative infinite, or, as it is often stated, the logarithm of zero is negative infinity.*

EXERCISE 36.

1. Find $\log_4 16$; $\log_4 64$; $\log_3 81$; $\log_4 \frac{1}{16}$; $\log_9 \frac{1}{81}$; $\log_3 \frac{1}{81}$; $\log_2 \frac{1}{32}$; $\log_5 \frac{1}{125}$; $\log_5 125$.

2. If 10 is the base, between what integral numbers does the logarithm of any number between 1 and 10 lie? Of any number between 10 and 100? Of any number between 100 and 1000? Of any number between 0.1 and 1? Of any number between 0.01 and 0.1? Of any number between 0.001 and 0.01?

In the next ten examples express $\log_a y$ in terms of $\log_a b$, $\log_a c$, $\log_a x$, and $\log_a z$.

3. $y = z^2 b^{\frac{3}{4}}$.

4. $y = \sqrt[3]{x^2} \cdot \sqrt{z^3}$.

5. $y = \sqrt[7]{z^3 x^4}$.

6. $y = \sqrt{z^3 x} \cdot \sqrt[3]{z b^{-5}}$.

7. $y = \dfrac{z^{-2}x^{\frac{3}{2}}}{b^m c^n}$. 8. $y = \dfrac{\sqrt[3]{xz^{-1}b^{-4}}}{(x^{-1}z^{-5}c^{\frac{3}{4}})^{\frac{1}{6}}}$.

9. $x^{\frac{1}{2}} : bz^{\frac{3}{4}} :: y^3 : xb^{\frac{2}{3}}$.

10. $x^{-\frac{3}{4}} : c^2y^{\frac{1}{2}} :: z^3 : x^2b^{\frac{1}{2}}$.

11. $\dfrac{z^4}{c^6} : \dfrac{x^{\frac{3}{4}}}{y^2} :: \dfrac{b^{-\frac{3}{4}}}{z^{\frac{4}{5}}} : \dfrac{\sqrt{z}}{\sqrt[3]{x^2}}$.

12. $z^n : \dfrac{x^{-\frac{1}{2}}}{b^{\frac{2}{3}}} :: (c^2b^{-\frac{1}{2}})^{\frac{1}{3}} : c^m y^{\frac{1}{2}}$.

Common Logarithms.

309. Although there may be any number of systems of logarithms, there are in general use only two, the *Napierian* and the *Common*. The *Napierian* system, so called from its inventor, Baron Napier, is used for analytical purposes only; its base is 2.71828. The *Common* system is the system used in practical computations; its base is 10. It was introduced in 1615 by Briggs, a contemporary of Napier.

Both Napierian and common logarithms are written decimally. Hereafter when no base is written, the base 10 is understood.

310. From the equation $10^x = N$, it is evident that the common logarithms of most numbers consist of an integral part and a fractional part.

For example, $2146 > 10^3$ and $< 10^4$;

$\therefore$ $\log 2146 = 3 +$ a decimal fraction.

Again, $0.04 > 10^{-2}$ and $< 10^{-1}$;

$\therefore$ $\log 0.04 = -2 +$ a decimal.

The integral part of a logarithm is called the **Characteristic**, and the decimal part the **Mantissa.** For convenience in the use of common logarithms, mantissas are always made positive. Hence the logarithm of any number less than unity consists of a *negative characteristic* and a *positive mantissa.*

311. The characteristic of the common logarithm of any number can be determined by one of the two following simple rules:

(i.) *If the number is greater than unity, the characteristic is positive and numerically one less than the number of digits in its integral part.*

For a number with one digit in its integral part lies between 10^0 and 10^1; a number with two digits in its integral part lies between 10^1 and 10^2; and so on. Hence if N denote a number that has n digits in its integral part, then N lies between 10^{n-1} and 10^n; that is,

$$N = 10^{(n-1) + \text{a fraction}}.$$

$$\therefore \log N = (n-1) + \text{a mantissa}.$$

Thus, $\log 2178.24 = 3 +$ a mantissa;

$\log 3872416 = 6 +$ a mantissa.

(ii.) *If the number is less than unity, the characteristic is negative and numerically one greater than the number of ciphers immediately after the decimal point.*

For a decimal with no cipher immediately after the decimal point lies between 10^{-1} and 10^0; thus, 0.327 lies between 0.1 and 1; a decimal with one cipher immediately after the decimal point lies between 10^{-2} and 10^{-1}; thus, 0.0217 lies between 0.01 and 0.1; and so on. Hence if D denote a decimal with n ciphers immediately after the decimal point, then D lies between $10^{-(n+1)}$ and 10^{-n}; that is,

$$D = 10^{-(n+1)+\text{a fraction.}}$$

$$\therefore \log D = -(n+1) + \text{a mantissa.}$$

Thus $\log 0.003217 = -3 + \text{a mantissa};$
$\log 0.000081 = -5 + \text{a mantissa}.$

The converse of rules (i.) and (ii.) may be stated as follows:

(i.) *If the characteristic of a logarithm is* + n, *there are* n + 1 *integral places in the corresponding number.*

(ii.) *If the characteristic is* − n, *there are* n − 1 *ciphers immediately to the right of the decimal point in the number.*

312. $\text{Log}\,(N \times 10^{\pm n}) = \log N \pm n.$ § 303.

Hence if n is a whole number, log N and log $(N \times 10^{\pm n})$ have the same mantissa. Therefore if

a number be multiplied or divided by an exact power of 10, the mantissa of its logarithm will not be changed.

That is, *the common logarithms of all numbers that have the same sequence of significant digits have the same mantissa.*

Thus, the logarithms of 21.78, 2178, and 0.002178 have the same mantissa.

313. The method of calculating logarithms will be explained in §§ 319, 322. The common logarithms of all integers from 1 to 200000 have been computed and tabulated. In most tables they are given to seven places of decimals; but in abridged tables they are often given to only four or five places. Common logarithms have two great practical advantages:

(i.) Characteristics are known by § 311, so that only mantissas are tabulated.

(ii.) Mantissas are determined by the sequence of digits (§ 312), so that the mantissas of integers only are tabulated.

When the characteristic is negative, the minus sign is written *over the characteristic*, to indicate that the characteristic alone is negative, and not the whole expression.

Thus $\bar{3}.845098$, the logarithm of 0.007, is equivalent to $-3 + 0.845098$, and must be distinguished from -3.845098, in which both the integral and decimal part are negative.

To transform a negative logarithm, as -3.26782, so that the mantissa shall be positive, we subtract 1 from the characteristic and add 1 to the mantissa.

Thus $-3.26782 = -4 + (1 - 0.26782) = \bar{4}.73218$.

To divide $\bar{3}.78542$ by 5, we proceed thus:

$$\tfrac{1}{5}(\bar{3}.78542) = \tfrac{1}{5}(-5 + 2.78542)$$
$$= \bar{1}.55708.$$

314. For logarithmic tables and directions in their use, the student is referred to works on Trigonometry. For use in this and the next chapter we give below the common logarithms of prime numbers from 1 to 100.

No.	Logarithms.	No.	Logarithms.	No.	Logarithms.
2	0.3010300	29	1.4623980	61	1.7853298
3	0.4771213	31	1.4913617	67	1.8260748
7	0.8450980	37	1.5682017	71	1.8512583
11	1.0413927	41	1.6127839	73	1.8633229
13	1.1139434	43	1.6334685	79	1.8976271
17	1.2304489	47	1.6720979	83	1.9190781
19	1.2787536	53	1.7242759	89	1.9493900
23	1.3617278	59	1.7708520	97	1.9867717

$$\begin{aligned}\text{Log}\,5 &= \log(10 \div 2)\\ &= \log 10 - \log 2 = 1 - 0.30103 = 0.69897.\end{aligned}$$

In like manner the logarithms of all integers between 1 and 100 can be obtained from those given in the table above.

The utility of logarithms in facilitating numerical computations is illustrated by the following example.

Example. Find the value of $3^{\frac{2}{3}} \times 0.9^2 \div 0.49^{\frac{3}{4}}$, given

$$\log 2.87686 = 0.458919.$$

$$\begin{aligned}\log\left(3^{\frac{2}{3}} \times 0.9^2 \div 0.49^{\frac{3}{4}}\right) &= \tfrac{2}{3}\log 3 + 2\log\tfrac{9}{10} - \tfrac{3}{4}\log\tfrac{49}{100}\\ &= \tfrac{2}{3}\log 3 + 2(\log 3^2 - 1) - \tfrac{3}{4}(\log 7^2 - 2)\\ &= \tfrac{2}{3}\log 3 + 4\log 3 - 2 - \tfrac{3}{2}\log 7 + \tfrac{3}{2}\\ &= \tfrac{14}{3}\log 3 - \tfrac{3}{2}\log 7 - \tfrac{1}{2}\\ &= 2.2265661 - 1.267647 - 0.5\\ &= 0.458919 = \log 2.87686;\end{aligned}$$

$$\therefore\ 3^{\frac{2}{3}} \times 0.9^2 \div 0.49^{\frac{3}{4}} = 2.87686.$$

EXERCISE 37.

1. Given $\log 2659 = 3.4224718$; find $\log 26.59$, $\log 0.2659$, $\log 265900$, $\log 0.0002659$.

2. Given $\log 2389 = 3.378216$; find the number whose logarithm is 1.378216, 0.378216, $\bar{2}.378216$, 5.378216, $\bar{3}.378216$, $\bar{4}.378216$.

Find the common logarithm of

3. 84.	8. 1.05.	13. $\sqrt[4]{0.0105}$.
4. 0.128.	9. 0.0183.	14. $86^{\frac{2}{3}}$.
5. 0.0125.	10. 0.02134.	15. $\sqrt{35 \div 27}$.
6. 1.44.	11. $\sqrt[3]{42}$.	16. $4\frac{2}{3}$.
7. 1.06.	12. $\sqrt{374}$.	17. $25^{\frac{2}{3}}$.

18. $2101^{\frac{9}{11}}$.

19. $0.015^{\frac{1}{5}}$.

20. $0.0018^{\frac{1}{7}}$.

21. $0.63^{\frac{1}{11}}$.

22. $(14 \div 15)^{\frac{11}{7}}$.

23. $\sqrt[8]{\dfrac{3^2 \times 5^4}{\sqrt{2}}}$.

24. $\dfrac{\sqrt[8]{48} \cdot \sqrt[4]{108}}{\sqrt[12]{6}}$.

25. $\dfrac{(336 \div 50)^4}{\sqrt[3]{24 \times 70}}$.

26. $\dfrac{(0.21 \div 0.07)^{\frac{1}{5}}}{(0.002 \div 3)^{\frac{4}{3}}}$.

27. Find the seventh root of 0.00324, given

$$\log 4409.2388 = 3.644363.$$

28. Find the eleventh root of 39.2^2, given

$$\log 19.48445 = 1.2896883.$$

29. Find the product of 37.203, 3.7203, 0.0037203, and 372030; given

$\log 372.03 = 2.570578$, and $\log 191.5631 = 2.282312$.

315. Exponential Equations. An *exponential* equation is one in which the unknown quantity appears in an exponent. Thus $2^x = 5$, $b^{2x} + b^x = c$, and $x^x = 10$ are exponential equations. Exponential equations are solved by the aid of logarithms.

EXAMPLE 1. Solve $3^{2x} 4^{3x} = 5^{4x} 2^{x+1}$.

Taking the logarithms of both members, we have

$$2x \log 3 + 3x \log 2^2 = 4x \log 5 + (x + 1) \log 2;$$

$$\therefore \ (2 \log 3 + 6 \log 2 - 4 \log 5 - \log 2)\, x = \log 2,$$

or

$$x = \frac{\log 2}{2 \log 3 + 5 \log 2 - 4 \log 5}$$

$$= \frac{0.301030}{-0.336488} = -0.894+.$$

EXAMPLE 2. Find the logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$.

Let $$x = \log 32\sqrt[5]{4} \text{ to base } 2\sqrt{2}, \text{ or } 2^{\frac{3}{2}};$$
then $$(2^{\frac{3}{2}})^x = 32\sqrt[5]{4} = 2^5 \times 2^{\frac{2}{5}}.$$

Hence $\frac{3}{2}x \log 2 = 5 \log 2 + \frac{2}{5} \log 2;$

$$\therefore \quad x = \frac{27}{5} \div \frac{3}{2} = \frac{18}{5} = 3.6.$$

EXAMPLE 3. Solve $3^{2x} - 14 \times 3^x + 45 = 0$.

The equation may be written in the form

$$(3^x - 9)(3^x - 5) = 0,$$

which is equivalent to the two equations

$$3^x = 9 \text{ and } 3^x = 5.$$

From $3^x = 9$, $x = 2$; and from $3^x = 5$,

$$x = \frac{\log 5}{\log 3} = \frac{0.698970}{0.477121} = 1.4649.$$

EXERCISE 38.

Solve the following literal equations:

1. $a^x = cb^x$.
2. $a^{2x} b^{3x} = c^5$.
3. $\dfrac{a^{x+1}}{b^{x-1}} = c^{2x}$.
4. $4b = c^{ax}$.

Solve the following numerical equations, using the table in § 314:

5. $5^x = 800$.
6. $5^{x-3} = 8^{2x+1}$.
7. $12^x = 3528$.
8. $2^{3x} 5^{2x-1} = 4^{5x} 3^{x+1}$.
9. $2^x 6^{x-2} = 5^{2x} 7^{1-x}$.
10. $4^{2x} + 56 = 15 \times 4^x$.

Find the logarithm of

11. 16 to base $\sqrt{2}$, and 1728 to base $2\sqrt{3}$.

12. 125 to base $5\sqrt{5}$, and 0.25 to base 4.

13. $\frac{1}{256}$ to base $2\sqrt{2}$, and 0.0625 to base 2.

14. Find $\log_8 128$; $\log_6 \frac{1}{216}$; $\log_{27} \frac{1}{81}$.

Solve the system of equations

15. $x^y = y^x$, $x^3 = y^2$.

16. $a^{2x} b^{3y} = m^5$, $a^{3x} b^{2y} = m^{10}$.

Logarithmic and Exponential Series.

316. The Differential of $\log_a y$. Let z denote any function of x, and let

$$y = n z, \qquad (1)$$

in which n is an arbitrary constant.

Then $$\log_a y = \log_a n + \log_a z;$$

$$\therefore\ d(\log_a y) = d(\log_a z). \qquad (2)$$

Differentiating (1), and dividing the result by (1),

$$\frac{dy}{y} = \frac{dz}{z}. \qquad (3)$$

Dividing (2) by (3), we obtain,

$$d(\log_a y) : \frac{dy}{y} = d(\log_a z) : \frac{dz}{z}. \qquad (4)$$

It is evident that the equal ratios in (4) are constant for any particular value of z. Let m denote their constant value when $z = z'$;

then $$d(\log_a y) = m \frac{dy}{y}, \tag{5}$$

when $y = nz'$. But as n is an arbitrary constant, nz' denotes any number; hence (5) holds true for all values of y, m being a constant.

The constant m is called the **Modulus** of the system of logarithms whose base is a.

Hence, *the differential of the logarithm of a variable is equal to the modulus of the system into the differential of the variable divided by the variable.*

EXERCISE 39.

Differentiate

1. $y = \log_a (1 + x)$.
2. $y = \log_a (x^3 + x^2)$.
3. $y = \log_a (x^2 + x + b)$.
4. $f(x) = (\log_a x)^3$.
5. $f(x) = \log_a x^3$.
6. $f(x) = x \log_a x$.
7. $y = \log_a \sqrt{1 - x^3} = \frac{1}{2} \log_a (1 - x^3)$.
8. $y = \log_a (x^2 + x)^{\frac{3}{2}}$.
9. $y = \log_a \frac{x}{\sqrt{1 + x^2}}$.
10. $y = \log_a (x^4 - cx^3)^{\frac{5}{6}}$.
11. $y = \log_a \frac{\sqrt{1 + x}}{\sqrt{1 - x}}$.

317. *To deduce the Logarithmic Series.*

To do this, we develop $\log_a (1 + x)$ by Maclaurin's formula,

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + f^{\text{IV}}(0)\frac{x^4}{\lfloor 4} + \cdots$$

Here $f(x) = \log_a (1 + x), \quad \therefore \quad f(0) = 0;$

$$f'(x) = \frac{m}{1+x}, \qquad \therefore \quad f'(0) = m;$$

$$f''(x) = -\frac{m}{(1+x)^2}, \quad \therefore \quad f''(0) = -m;$$

$$f'''(x) = \frac{2\,m}{(1+x)^3}, \qquad \therefore \quad f'''(0) = \lfloor 2\, m;$$

$$f^{\text{IV}}(x) = -\frac{\lfloor 3\, m}{(1+x)^4}, \quad \therefore \quad f^{\text{IV}}(0) = -\lfloor 3\, m;$$

$$\cdots \qquad \cdots \qquad \qquad \cdots \qquad \cdots$$

Substituting these values in the formula, we have

$$\log_a (1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right), \quad \text{(A)}$$

which is the general *logarithmic series.*

318. Napierian System. The system of logarithms whose modulus is *unity* is called the *Napierian* or *Natural* system. The symbol for the Napierian base is e.

If in (A) of § 317 we put $m = 1$ and $a = e$, we have

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \qquad \text{(B)}$$

which is the *Napierian* logarithmic series.

By Example 1 of § 286 the series in (B) is convergent only for values of x between -1 and $+1$; hence formula (B) cannot be used to compute the Napierian logarithm of any number greater than 2.

319. *To obtain a formula for computing a table of Napierian logarithms.*

Putting $-x$ for x in (B) of § 318, we have

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \qquad (1)$$

Subtracting (1) from (B), we have

$$\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right). \qquad (2)$$

Let
$$x = \frac{1}{2z+1}; \qquad (3)$$

then
$$\frac{1+x}{1-x} = \frac{z+1}{z}.$$

$$\therefore \log_e(1+x) - \log_e(1-x) = \log_e(z+1) - \log_e z. \qquad (4)$$

Substituting in (2) the values in (3) and (4), we obtain

$$\log_e(z+1) = \log_e z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right). \qquad (C)$$

Since, in (3), $x < 1$ for $z > 0$, the series in (C) is convergent for all positive values of z; hence $\log_e(z+1)$ can be readily computed when $\log_e z$ is known.

Example. Compute to six places of decimals $\log_e 2$, $\log_e 3$, $\log_e 4$, $\log_e 5$, $\log_e 10$.

Putting $z = 1$ in (C), we obtain, since $\log_e 1 = 0$,

$$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right).$$

Summing six terms of this series, we find

$$\log_e 2 = 0.693147.$$

Putting $z = 2$ in (C), we have

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \cdots\right)$$

$$= 1.098612.$$

$\text{Log}_e 4 = 2 \log_e 2 = 1.386294.$

Putting $z = 4$ in (C), we obtain

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \cdots\right)$$

$$= 1.6094379.$$

$\text{Log}_e 10 = \log_e 5 + \log_e 2 = 2\ 302585.$

In this way the Napierian logarithms of all positive numbers can be computed. The larger the number the more rapidly convergent is the series.

320. Value of m. Dividing (A) by (B), we have

$$\frac{\log_a (1 + x)}{\log_e (1 + x)} = m, \qquad (1)$$

in which $1 + x$ lies between 0 and 2.

Let N be any number, and let

$$y = \log_a N, \text{ or } N = a^y;$$

then $$\log_e N = y \cdot \log_e a = \log_a N \cdot \log_e a.$$

Hence $$\frac{\log_a N}{\log_e N} = \frac{1}{\log_e a}, \text{ a constant.} \tag{2}$$

Let $N = 1 + x$; then from (1) and (2), we have

$$m = \frac{1}{\log_e a}. \tag{3}$$

That is, *the modulus of any system of logarithms is equal to the reciprocal of the Napierian logarithm of its base.*

321. From (2) and (3) of § 320 we have

$$\log_a N = m \log_e N.$$

That is, *the logarithm of a number in any system is equal to the Napierian logarithm of the same number multiplied by the modulus of that system.*

322. Value of M. If in (3) of § 320, M denote the value of m when $a = 10$, we obtain

$$M = \frac{1}{\log_e 10} = \frac{1}{2.302585} = 0.434294.$$

That is, *the modulus of the Common System to six places of decimals is* 0.434294.

Hence to obtain common logarithms from Napierian, multiply the Napierian by 0.434294; to obtain Napierian from common, multiply the common by 2.302585, or $\log_e 10$.

Multiplying both members of (C) by M, we obtain

$$\log(z+1)=\log z+2M\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\cdots\right),$$

which is a formula for computing common logarithms.

Multiplying both numbers of (C) by m, we obtain a general formula for computing logarithms to any base a.

323. An **Exponential Function** is one in which the variable enters the exponent, as a^x, y^x, a^{b+cx}.

324. *To differentiate the exponential function* a .

Let $\qquad y=a^x;$ then $\log_e y = x\log_e a. \qquad (1)$

Hence $$\frac{dy}{y}=\log_e a\cdot dx,$$

or $$dy=d(a^x)=a^x\log_e a\cdot dx.$$

That is, *the differential of an exponential function with a constant base is equal to the function itself into the Napierian logarithm of the base into the differential of the exponent.*

325. *To develop* a^x, *or deduce the Exponential Series.*

Here
$$\begin{aligned}
f(x)&=a^x, & \therefore\ f(0)&=1;\\
f'(x)&=a^x\log_e a, & \therefore\ f'(0)&=\log_e a;\\
f''(x)&=a^x(\log_e a)^2, & \therefore\ f''(0)&=(\log_e a)^2;\\
f'''(x)&=a^x(\log_e a)^3, & \therefore\ f'''(0)&=(\log_e a)^3;\\
\cdots\quad & \quad\cdots & \cdots\quad & \quad\cdots
\end{aligned}$$

Substituting these values in Maclaurin's formula, we have

$$a^x = 1 + (\log_e a)\, x + (\log_e a)^2 \frac{x^2}{\lfloor 2} + (\log_e a)^3 \frac{x^3}{\lfloor 3} + \cdots, \quad (1)$$

which is the *exponential series*.

326. Value of e^x. Putting $a = e$ in (1) of § 325, we have

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots \quad (1)$$

327. Value of e. Putting $x = 1$ in (1) of § 326, we obtain

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots = 2.718281.$$

That is, *the Napierian base* $= 2.718281 +$.*

EXERCISE 40.

1. Find to five places $\log_e 6$, $\log_e 7$, $\log_e 8$, $\log_e 9$, $\log_e 11$.

2. Find to five places the moduli of the systems whose bases are 2, 3, 4, 5, 8, 9, 12.

3. By § 321, prove that the logarithms of the same number in different systems are proportional to the moduli of those systems.

* In the "Proceedings of the Royal Society of London," Vol. XXVII., Prof. J. C. Adams has given the values of e, M, $\log_e 2$, $\log_e 3$, $\log_e 5$, to more than 260 places of decimals.

4. By the formula of § 322 compute log 2, log 65, log 131, log 3, log 82, log 244.

5. Obtain the formula,

$$\log_a (z + 1) = \log_a z + 2m \left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots \right).$$

6. Obtain the following formulas:

$$\log_e (z + 1) - \log_e z = \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \cdots \qquad (1)$$

$$\log_e z - \log_e (z - 1) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots \qquad (2)$$

$$\log_e (z + 1) - \log_e (z - 1) = 2\left(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \cdots \right). \qquad (3)$$

These formulas are convergent for $z > 1$, and may be used in computing logarithms.

To obtain (1), in (B) substitute $(1 \div z)$ for x; to obtain (2), substitute $(-1 \div z)$ for x.

7. Obtain the formulas corresponding to (1), (2), and (3) of Example 6, for common logarithms; for any system.

8. Show that $\log \dfrac{\sqrt[4]{5} \cdot \sqrt[10]{2}}{\sqrt[3]{18} \cdot \sqrt{2}} = \frac{1}{4} \log 5 - \frac{2}{5} \log 2 - \frac{2}{3} \log 3.$

9. Show that $\log \sqrt[4]{729 \sqrt[3]{9^{-1} \times 27^{-\frac{4}{3}}}} = \log 3.$

10. Find the logarithms of $\sqrt[4]{a^{\frac{8}{5}}}, \dfrac{1}{\sqrt{a}}, \sqrt[8]{a^{-\frac{15}{2}}}$, to base a.

11. Find the number of digits in $3^{12} \times 2^8$.

12. Show that $\left(\frac{21}{20}\right)^{100}$ is greater than 100.

13. Find how many ciphers there are between the decimal point and the first significant digit in $(\frac{1}{3})^{1000}$.

14. Calculate to six decimal places the value of

$$\sqrt[8]{\left(\frac{294 \times 125}{42 \times 32}\right)^2};$$

given $\log 9076.226 = 3.9579053$.

Solve

15. $2^{x+y} = 6^y,$
$3^x = 3 \times 2^{y+1}.$

16. $3^{1-x-y} = 4^{-y},$
$2^{2x-1} = 3^{3y-x}.$

17. If $\log x^2 y^3 = a$, and $\log (x - y) = b$, find $\log x$ and $\log y$.

18. Show that $\underset{m=\infty}{\text{limit}} \left(1 + \frac{x}{m}\right)^m = e^x$.

$$\left(1 + \frac{x}{m}\right)^m = 1 + m\frac{x}{m} + \frac{m(m-1)}{\lfloor 2}\left(\frac{x}{m}\right)^2 + \cdots$$

$$= 1 + x + \frac{1 - \frac{1}{m}}{\lfloor 2}x^2 + \frac{\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{\lfloor 3}x^3 + \cdots;$$

$$\therefore \quad \underset{m=\infty}{\text{limit}} \left(1 + \frac{x}{m}\right)^m = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots = e^x.$$

19. If a is positive, the positive real value of a^x is a continuous function of x.

For a^x has one positive real value for each value of x, and

$$\Delta(a^x) = a^x(a^{\Delta x} - 1) \doteq 0 \text{ when } \Delta x \doteq 0.$$

CHAPTER XVII.

COMPOUND INTEREST AND ANNUITIES.

328. *To find the interest* I *and amount* M *of a given sum* P *in* n *years at* r *per cent, compound interest.*

(i.) *When interest is payable annually.*

Let $R =$ the amount of \$1 in 1 year; then $R = 1 + r$, and the amount of P at the end of the first year is PR; and since this is the principal for the second year, the amount at the end of the second year is $PR \times R$, or PR^2. For like reason the amount at the end of the third year is PR^3, and so on; hence the amount in n years is PR^n; that is

$$M = PR^n, \text{ or } P(1 + r)^n. \qquad (1)$$

Hence

$$I = P(R^n - 1). \qquad (2)$$

(ii.) *When the interest is payable* q *times a year.*

If the interest is payable semi-annually, then the interest of \$1 for $\frac{1}{2}$ a year is $\frac{1}{2}r$; hence

the amount of P in $\frac{1}{2}$ a year is $P(1 + \frac{1}{2}r)$;

the amount of P in one year is $P(1 + \frac{1}{2}r)^2$;

the amount of P in n years is $P(1 + \frac{1}{2}r)^{2n}$.

That is, $$M = P(1 + \tfrac{1}{2}r)^{2n}. \tag{3}$$

Similarly, if the interest is payable quarterly,

$$M = P(1 + \tfrac{1}{4}r)^{4n}. \tag{4}$$

Hence, if the interest is payable q times a year,

$$M = P\left(1 + \frac{1}{q}r\right)^{qn}. \tag{5}$$

In this case the interest is said to be "converted into principal" q times a year.

EXAMPLE. Find the time in which a sum of money will double itself at 10 per cent compound interest, interest payable semi-annually.

Here $1 + \frac{1}{2}r = 1.05$.

Let $P = \$1$; then $M = \$2$.

Substituting these values in (3), we obtain

$$2 = (1.05)^{2n};$$

$$\therefore\ \log 2 = 2n \cdot \log 1.05;$$

$$\therefore\ n = \frac{\log 2}{2(\log 5 + \log 3 + \log 7 - 2)}$$

$$= \frac{0.30103}{2(0.69897 + 0.4771213 + 0.845098 - 2)}$$

$$= 7.103.$$

Therefore the time is 7.103 years.

329. When the time contains a fraction of a year, it is usual to allow simple interest for the fraction of the year. Thus the amount of P in $n + \frac{1}{m}$ years is

$$PR^n + PR^n \frac{r}{m}, \text{ or } PR^n\left(1 + \frac{r}{m}\right).$$

When interest is payable oftener than once a year there is a difference between the *nominal annual rate* and the *true annual rate.* Thus, if interest is payable semi-annually at the nominal annual rate r, the amount of \$1 in one year is $(1 + \frac{1}{2}r)^2$, or $1 + r + \frac{1}{4}r^2$, so that the true annual rate is $r + \frac{1}{4}r^2$.

Thus, if the nominal annual rate is 4 per cent, and interest is payable semi-annually, the true annual rate is 4.04 per cent.

330. Present Value and Discount. Let P denote the present value of the sum M due in n years, at the rate r; then evidently, in n years, at the rate r, P will amount to M; hence

$$M = PR^n, \text{ or } P = MR^{-n}.$$

Let D be the discount; then

$$D = M - P = M(1 - R^{-n}).$$

EXERCISE 41.

1. Write out the logarithmic equations for finding each of the four quantities M, R, P, n.

2. In what time, at 5 per cent compound interest, will \$100 amount to \$1000?

3. Find the time in which a sum will double itself at 4 per cent compound interest.

4. Find in how many years \$1000 will become \$2500 at 10 per cent compound interest.

5. Find the present value of \$10,000 due 8 years hence at 5 per cent compound interest; given $\log 67683.94 = 4.8304856$.

6. Find the amount of \$1 at 5 per cent compound interest in a century; given $\log 1315 = 3.11893$.

7. Show that money will increase more than seventeen-thousand-fold in a century at 10 per cent compound interest, interest payable semi-annually; given $\log 17213.13 = 4.23786$.

8. Find what sum of money at 6 per cent compound interest will amount to \$1000 in 12 years; given $\log 49697 = 4.6963292$, $\log 106 = 2.0253059$.

9. Find the amount of a cent in 200 years at 6 per cent compound interest; given $\log 115.128 = 2.06118$.

10. The present value of \$672 due in a certain time is \$126; if compound interest at $4\frac{1}{2}$ per cent be allowed, find the time.

ANNUITIES.

331. An **Annuity** is a fixed sum of money that is payable once a year, or at more frequent regular intervals, under certain stated conditions. An *Annuity Certain* is one payable for a fixed number of years. A *Life Annuity* is one payable during the

lifetime of a person. A *Perpetual Annuity*, or *Perpetuity*, is one that is to continue forever, as, for instance, the rent of a freehold estate. A *Deferred Annuity* is one that does not begin until after a certain number of years.

332. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let A be the annuity, n the number of years, R the amount of one dollar in one year, M the required amount. Then evidently the sum due at the end of the

$$\begin{aligned} \text{1st year} &= A. \\ \text{2d year} &= AR + A. \\ \text{3d year} &= AR^2 + AR + A. \\ n\text{th year} &= AR^{n-1} + AR^{n-2} + \cdots + AR + A \\ &= \frac{A(R^n - 1)}{R - 1}. \end{aligned}$$

That is, $$M = \frac{A}{r}(R^n - 1). \qquad (1)$$

EXAMPLE 1. Find the amount of an annuity of \$100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent; given $\log 1.045 = 0.0191163$, $\log 24.117 = 1.382326$.

$$M = \frac{A}{r}(R^n - 1) = \frac{\$100\,(1.045^{20} - 1)}{0.045}.$$

By logarithms $1.045^{20} = 2.4117$;

$$\therefore M = \frac{\$141.17}{0.045} = \$3137.11.$$

Example 2. Find what sum must be set aside annually that it may amount to \$50,000 in 10 years at 6 per cent compound interest; given log $17.9085 = 1.253059$.

Solving (1) for A we obtain

$$A = \frac{Mr}{R^n - 1} = \frac{\$50,000 \times 0.06}{1.06^{10} - 1}.$$

By logarithms $\quad 1.06^{10} = 1.79085;$

$$\therefore \quad A = \frac{\$3000}{0.79085} = \$3793.37.$$

333. *To find the present value of an annuity to continue for a given number of years, allowing compound interest.*

Let P denote the present value; then the amount of P in n years will equal the amount of the annuity in the same time; that is,

$$PR^n = \frac{A}{r}(R^n - 1); \qquad (1)$$

$$\therefore P = \frac{A}{r}(1 - R^{-n}). \qquad (2)$$

334. Perpetuity. If the annuity be perpetual, then $n = \infty$, $R^{-n} \doteq 0$, and (2) of § 333 becomes

$$P = \frac{A}{r}.$$

335. Deferred Annuity. If the annuity commences after p years, and continues n years thereafter, then the present value will evidently be the difference

between the present value of an annuity to continue $n+p$ years and one to continue p years; that is,

$$P = \frac{A}{r}(R^{-p} - R^{-n-p}), \qquad (1)$$

or

$$P = \frac{A}{r} \times \frac{R^n - 1}{R^{n+p}}.$$

336. If the annuity be perpetual after p years, then $R^{-n-p} \doteq 0$, and (1) of § 335 becomes

$$P = \frac{A}{r} R^{-p}.$$

337. Solving (1) of § 333 for A, we obtain

$$A = \frac{Pr R^n}{R^n - 1},$$

which gives the value of the annuity in terms of the present value, the time, and the rate per cent.

338. A **Freehold Estate** is an estate which yields a perpetual annuity, called rent; hence the value of the estate is the present value of a perpetuity equal to the rent.

EXAMPLE 1. Find the present value of an annual pension of \$200 for 10 years at 5 per cent interest; given log 6.13917 = 0.788107.

$$P = \frac{A}{r}(1 - R^{-n}) = \frac{\$200}{0.05}(1 - 1.05^{-10}).$$

By logarithms $1.05^{-10} = 0.613917$;

$$\therefore \quad P = \$4000 \times 0.386083 = \$1544.33.$$

EXAMPLE 2. The rent of a freehold estate is \$350 a year. Find the value of the estate, the rate of interest being 5 per cent.

$$P = \frac{A}{r} = \frac{\$350}{0.05} = \$7000.$$

EXAMPLE 3. Find the present value of an annuity of \$1400 to begin in 8 years and to continue 12 years, at 8 per cent interest; given $\log 25.1818 = 1.4010868$, $\log 466.1 = 2.668478$.

$$P = \frac{A}{r} \times \frac{R^n - 1}{R^{n+p}} = \frac{\$1400}{0.08} \times \frac{1.08^{12} - 1}{1.08^{20}} = \$5700.09.$$

EXAMPLE 4. Find what annuity \$5000 will give for 6 years when money is worth 6 per cent; given $\log 14.185 = 1.1518344$.

$$A = \frac{PrR^n}{R^n - 1} = \$5000 \times 0.06 \times \frac{1.06^6}{1.06^6 - 1} = \$1016.84.$$

EXERCISE 42.

1. If A leaves B \$1000 a year to accumulate for 3 years at 4 per cent compound interest, find what amount B should receive; given $\log 112.4864 = 2.0511002$.

2. Find the present value of the legacy in Example 1; given $\log 888.9955 = 2.9488998$.

3. Find the present value, at 5 per cent, of an estate of \$1000 a year, (1) to be entered on immediately, (2) after 3 years; given $\log 17276.75 = 4.2374621$.

4. A freehold estate worth \$120 a year is sold for \$4000; find the rate of interest.

5. A man borrows \$5000 at 4 per cent compound interest; if the principal and interest are to be repaid by 10 equal annual instalments, find the amount of each instalment; given $\log 676031 = 5.829667$.

6. A man has a capital of \$20,000, for which he receives interest at 5 per cent; if he spends \$1800 every year, show that he will be ruined before the end of the 17th year.

7. When the rate of interest is 4 per cent, find what sum must be paid now to receive a freehold estate of \$400 a year 10 years hence; given $\log 6.75565 = 0.829667$.

8. The rent of a freehold estate of \$882 per annum, deferred for two years, is to be sold; find its present value at 5 per cent compound interest.

9. The rent of a freehold estate, deferred for 6 years, is bought for \$20,000; find what rent the purchaser should receive, reckoning compound interest at 5 per cent; given $\log 1.340096 = 0.1271358$.

10. Find the present worth of a perpetual annuity of \$10 payable at the end of the first year, \$20 at the end of the second, \$30 at the end of the third, and so on, increasing \$10 each year, interest being taken at 5 per cent per annum.

CHAPTER XVIII.

PERMUTATIONS AND COMBINATIONS.

339. Fundamental Principle. *If one thing can be done in* m *ways, and (after it has been done in any one of these ways) a second thing can be done in* n *ways; then the two things can be done in* $m \times n$ *ways.*

After the first thing has been done in *any one* way, the second thing can be done in n different ways; hence there are n ways of doing the two things for *each* of the m ways of doing the first; therefore in all there are mn ways of doing the two things.

This principle is readily extended to the case in which there are three or more things, each of which can be done in a given number of ways.

EXAMPLE 1. If there are 11 steamers plying between New York and Havana, in how many ways can a man go from New York to Havana and return by a different steamer?

He can make the first passage in 11 ways, with each of which he has the choice of 10 ways of returning; hence he can make the two journeys in 11×10, or 110, ways.

EXAMPLE 2. In how many ways can 3 prizes be given to a class of 10 boys, without giving more than one to the same boy?

The first prize can be given in 10 ways; with each of which the second prize can be given in 9 ways; hence the first two prizes can be given in 10×9 ways. With each of these ways the third prize can be given in 8 ways; hence the three prizes can be given in $10 \times 9 \times 8$, or 720, ways.

340. Each of the different groups of r things which can be made of n things is called a **Combination.**

The **Permutations** of any number of things are the different orders in which they can be arranged, taking a certain number at a time.

Thus of the four letters a, b, c, d, taken two at a time, there are six combinations; namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd.$$

Each of these groups can be arranged in two different orders; hence of the four letters a, b, c, d, taken two at a time there are twelve permutations; namely,

$$\begin{array}{cccccc} ab, & ac, & ad, & bc, & bd, & cd, \\ ba, & ca, & da, & cb, & db, & dc. \end{array}$$

Of a group of three letters, as abc, when taken all at a time, there are six permutations; namely,

$$abc, \quad acb, \quad bca, \quad bac, \quad cab, \quad cba.$$

The symbol ${}^nP_r$ will be used to denote the number of permutations of n things taken r at a time.

Thus, ${}^9P_2$, ${}^9P_3$, ${}^9P_4$, denote respectively the number of permutations of 9 things taken 2, 3, 4, at a time.

Similarly ${}^nC_r$ will be used to denote the number of combinations of n things taken r at a time.

341. *To find the number of permutations of* n *dissimilar things taken* r *at a time.*

The number required is the same as the number of ways of filling r places with n things.

Now, the first place can be filled by any one of the n things, and after this has been filled in any one of these n ways, the second place can evidently be filled in $(n-1)$ ways; hence with n things two places can be filled in $n(n-1)$ ways; that is,

$$^nP_2 = n(n-1). \tag{1}$$

After the first two places have been filled in any one of these $n(n-1)$ ways, the third place can be filled in $(n-2)$ ways; hence three places can be filled in $n(n-1)(n-2)$ ways; that is,

$$^nP_3 = n(n-1)(n-2). \tag{2}$$

For like reason we have

$$^nP_4 = n(n-1)(n-2)(n-3); \tag{3}$$

and so on.

From (1), (2), (3), $\cdots$, we see that in ^nP_r there are r factors, of which the rth is $n-r+1$; hence

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1). \tag{A}$$

342. Value of ^nP_n. If $r=n$, (A) of § 341 becomes

$$^nP_n = \underline{|n}. \tag{B}$$

That is, *the number of permutations of* n *things taken all at a time is* $\underline{|n}$.

343. Circular Permutations. When n different letters are arranged in a circle, any one of their permutations can without change be revolved so that any letter, as a, shall have a given position. Hence we may regard a as having the same position in all the permutations. Now the number of the possible arrangements of the remaining $n-1$ letters in the other positions is $\lfloor n-1$.

Hence, *the number of the Circular Permutations of* n *things is* $\lfloor n-1$.

EXERCISE 43.

1. A cabinet-maker has 12 patterns of chairs and 7 patterns of tables. In how many ways can he make a chair and a table? Ans. 84.

2. There are 9 candidates for a classical, 8 for a mathematical, and 5 for a natural-science scholarship. In how many ways can the scholarships be awarded?

3. In how many ways can 2 prizes be awarded to a class of 10 boys, if both prizes may be given to the same boy?

4. Find the number of the permutations of the letters in the word *numbers*. How many of these begin with n and end with s?

5. If no digit occur more than once in the same number, how many different numbers can be represented by the 9 digits, taken 2 at a time? 3 at a time? 4 at a time?

6. How many changes can be rung with 5 bells out of 8? How many with the whole peal?

The first number $= {}^8P_5 = 6720$.

7. How many changes can be rung with 6 bells, the same bell always being last?

8. In how many ways may a host and 6 guests be seated at a table in a row? In how many ways if the host must have Mr. Jones on his right and Mr. Smith on his left? In how many ways if the host must sit between Mr. Smith and Mr. Jones?

9. In how many ways may 15 books be arranged on a shelf, the places of 2 being fixed?

10. Given ${}^nP_4 = 12 \cdot {}^nP_2$; find n.

11. Given $n : {}^nP_3 :: 1 : 20$; find n.

12. In how many different orders may a party of 6 be seated at a round table?

13. In how many different orders may 10 persons form a ring?

14. In how many different orders may a host and 8 guests sit at a round table, provided the host has Mr. A at his right and Mr. B at his left?

15. Given ${}^nP_3 : {}^{n+2}P_3 :: 5 : 12$, to find n.

16. Given ${}^nP_4 : {}^{\frac{2}{3}n}P_4 :: 13 : 2$, to find n.

344. *To find the number of combinations of* n *dissimilar things taken* r *at a time.*

By § 342 there are $\underline{|r}$ permutations of any combination of r things; hence we have

$$\begin{aligned} {}^nC_r \, \underline{|r} &= {}^nP_r \\ &= n(n-1)(n-2)\cdots(n-r+1). \end{aligned}$$

Hence $${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}. \qquad \text{(C)}$$

345. COROLLARY 1. Multiplying the numerator and denominator of the fraction in (C) by $\underline{|n-r}$, we obtain

$${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)\,\underline{|n-r}}{\underline{|r}\,\underline{|n-r}},$$

or $${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. \qquad \text{(D)}$$

Formula (C) should be used when a numerical result is required. In applying this formula, it is useful to note that the suffix r in the symbol ${}^nC_r$ denotes the number of the factors in both the numerator and denominator of the formula. Formula (D) gives the simplest algebraic expression for ${}^nC_r$.

346. COROLLARY 2. Substituting $n-r$ for r in (D) we obtain

$${}^nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}. \qquad (1)$$

From (D) and (1), ${}^nC_r = {}^nC_{n-r}$. (E)

The relation in (E) follows also from the consideration that for each group of r things that is selected, there is left a corresponding group of $n - r$ things. This relation often enables us to abridge arithmetical work.

Thus, $$^{15}C_{13} = {}^{15}C_2 = \frac{15 \times 14}{2} = 105.$$

EXERCISE 44.

1. How many combinations can be made of 9 things taken 4 at a time? taken 6 at a time? taken 7 at a time?

The last number $= {}^9C_7 = {}^9C_2 = 36$.

2. How many combinations can be made of 11 things taken 4 at a time? taken 7 at a time?

3. Out of 10 persons 4 are to be chosen by lot. In how many ways can this be done? In all the ways, how often would any one person be chosen?

4. From 14 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

5. On how many days might a person having 15 friends invite a different party of 10? of 12?

6. Given $\frac{1}{3}{}^nC_2 = 15$, to find n.

7. Given ${}^{n+1}C_4 = 9 \times {}^nC_2$, to find n.

8. In a certain district there are 4 representatives to be elected, and there are 7 candidates. How many different tickets can be made up?

9. Of 8 chemical elements that will unite with one another, how many ternary compounds can be formed? How many binary?

10. On a table are 6 Latin, 7 Greek, and 8 German books. In how many different ways may 2 books from different languages be chosen? In how many ways may 3?

The first number $= 6 \times 7 + 6 \times 8 + 7 \times 8 = 146$.

11. In how many ways can 10 gentlemen and 10 ladies arrange themselves in couples?

12. How many different arrangements of 6 letters can be made of the 26 letters of the alphabet, 2 of the 5 vowels being in every arrangement?

13. How many different straight lines can be drawn through any 15 points, no 3 of which lie in the same straight line?

14. In a town council there are 25 councillors and 10 aldermen; how many committees can be formed, each consisting of 5 councillors and 3 aldermen?

15. Find the sum of the products of the numbers 3, -2, 4, -5, 1, (1) taken 2 at a time, (2) taken 3 at a time, (3) taken 4 at a time.

16. Find the sum of the products of the numbers 1, 3, 5, 2, (1) taken 2 at a time, (2) taken 3 at a time.

17. Find the number of combinations of 55 things taken 45 at a time.

18. If ${}^{2n}C_3 : {}^{n}C_2 = 44 : 3$; find n.

19. If ${}^nC_{12} = {}^nC_8$; find ${}^nC_{17}$; find ${}^{22}C_n$.

20. In a library there are 20 Latin and 6 Greek books; in how many ways can a group of 5 consisting of 3 Latin and 2 Greek books be placed on a shelf?

21. From 3 capitals, 5 other consonants, and 4 other vowels, how many permutations can be made, each beginning with a capital and containing in addition 3 consonants and 2 vowels?

22. If ${}^{18}C_r = {}^{18}C_{r+2}$; find ${}^rC_5$.

23. From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done when the committee contains, (1) exactly 2 Americans, (2) at least 2 Americans?

24. Of 7 consonants and 4 vowels, how many permutations can be made, each containing 3 consonants and 2 vowels?

25. How many different arrangements can be made of the letters in the word *courage*, so that the consonants may occupy even places?

347. *If* N *denote the number of permutations of* n *things taken all at a time, of which* r *things are alike,* s *others alike, and* t *others alike; then*

$$N = \frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}}.$$

Suppose that in any one of the N permutations we replaced the r like things by r dissimilar things; then,

from this single permutation, without changing in it the position of any one of the other $n - r$ things, we could form $|\underline{r}$ new permutations. Hence from the N original permutations we could obtain $N|\underline{r}$ permutations, in each of which s things would be alike and t others alike.

Similarly, if the s like things were replaced by s dissimilar things, the number of permutations would be $N|\underline{r}\,|\underline{s}$, each having t things alike. Finally, if the t like things were replaced by t dissimilar things we should obtain $N|\underline{r}\,|\underline{s}\,|\underline{t}$ permutations, in which all the things would be dissimilar.

But the number of permutations of n dissimilar things taken all at a time is $|\underline{n}$.

Hence $$N|\underline{r}\,|\underline{s}\,|\underline{t} = |\underline{n}.$$

Therefore $$N = \frac{|\underline{n}}{|\underline{r}\,|\underline{s}\,|\underline{t}}.$$

348. *To find the number of ways in which* m + n *things can be divided into two groups containing respectively* m *and* n *things.*

The number required is evidently the same as the number of combinations of $m + n$ things taken m at a time; for every time a group of m things is selected a group of n things is left.

Hence the required number $= \dfrac{|\underline{m+n}}{|\underline{m}\,|\underline{n}}$. § 345.

349. By § 348 the number of ways in which $m+n+p$ things can be divided into two groups containing respectively m and $m+p$ things is

$$\frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n+p}}.$$

Again, the number of ways in which each group of $n+p$ things can be divided into two groups containing respectively n and p things is

$$\frac{\underline{|n+p}}{\underline{|n}\,\underline{|p}}.$$

Hence the number of ways in which $m+n+p$ things can be divided into three groups containing respectively m, n, and p things is

$$\frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n+p}} \times \frac{\underline{|n+p}}{\underline{|n}\,\underline{|p}}, \quad \text{or} \quad \frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n}\,\underline{|p}}.$$

This reasoning can be extended to any number of groups.

***350.** *The sum of all the combinations that can be made of* n *things, taken* 1, 2, ..., n *at a time, is* $2^n - 1$.

By the binomial theorem we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots \quad (1)$$

In (1) the coefficients of $x, x^2, x^3, \ldots, x^n$ are evidently the values of ${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n$; hence (1) may be written

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n. \quad (2)$$

Putting $x = 1$, and transposing 1, we obtain

$$2^n - 1 = {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n,$$

which proves the proposition.

***351.** ${}^nC_r$ *is greatest when* $r = \frac{1}{2}n$ *or* $r = \frac{1}{2}(n \pm 1)$, *according as* n *is even or odd.*

Evidently ${}^nC_r$, or $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$, is greatest when $\underline{|r}\,\underline{|n-r}$ is least.

Since $\underline{|a+1}\,\underline{|a-1}$ is obtained from $\underline{|a}\,\underline{|a}$ by multiplying by $a + 1$ and dividing by a, it follows that

$$\underline{|a}\,\underline{|a} < \underline{|a+1}\,\underline{|a-1} < \underline{|a+2}\,\underline{|a-2} < \cdots$$

Hence when n is even, $\underline{|r}\,\underline{|n-r}$ is least, and therefore ${}^nC_r$ is greatest, when $r = n - r$, or $r = \frac{1}{2}n$.

Again $\underline{|b}\,\underline{|b+1} = \underline{|b+1}\,\underline{|b}$,

and $\underline{|b+1}\,\underline{|b} < \underline{|b+2}\,\underline{|b-1} < \underline{|b+3}\,\underline{|b-2} < \cdots$

Hence when n is odd, $\underline{|r}\,\underline{|n-r}$ is least, and therefore ${}^nC_r$ is greatest, when $r = n - r \pm 1$, or $r = \frac{1}{2}(n \pm 1)$.

EXERCISE 45.

1. How many different arrangements can be made of the letters of the word *commencement*?

Of the 12 letters, 2 are c's, 3 are m's, 3 are e's, and 2 are n's;

$$\therefore N = \frac{\underline{|12}}{\underline{|2}\,\underline{|3}\,\underline{|3}\,\underline{|2}} = 3326400.$$

2. Find the number of permutations of the letters of the words, *mammalia*, *caravansera*, *Mississippi*.

3. In how many ways can 17 balls be arranged, if 7 of them are black, 6 red, and 4 white?

4. When repetitions are allowed, ${}^nP_r = n^r$, and ${}^nP_n = n^n$.

When repetitions are allowed after the first place has been filled in any one of n ways, the second place can be filled in n ways; hence ${}^nP_2 = n^2$, etc.

5. In how many ways can 4 prizes be awarded to 10 boys, each boy being eligible for all the prizes?

6. At an election three districts are to be canvassed by 10, 15, and 20 men, respectively. If 45 men volunteer, in how many ways can they be allotted to the different districts?

7. In how many ways can 52 cards be divided equally among 4 players?

8. In how many ways can mn things be divided equally among n persons?

9. How many signals can be made by hoisting 4 flags of different colors one above the other, when any number of them may be hoisted at once? How many with 5 flags?

10. There are 25 points in space, no 4 of which lie in the same plane. Find how many planes there are, each containing 3 points.

11. For what value of r is ${}^{10}C_r$ greatest? ${}^{11}C_r$? ${}^{15}C_r$? ${}^{20}C_r$? ${}^{81}C_r$?

12. For what value of r is $\underline{|r}\,\underline{|18 - r}$ least? $\underline{|r}\,\underline{|21 - r}$? $\underline{|r}\,\underline{|45 - r}$?

13. What term has the greatest coefficient in the development of $(x + y)^{14}$? $(x + y)^{17}$? $(x + y)^{21}$?

CHAPTER XIX.

PROBABILITY.

352. If an event may happen in a ways and fail in b ways, and each of these ways is equally likely, the **Probability**, or the **Chance**, of its happening is $\frac{a}{a+b}$, and the *probability* of its failing is $\frac{b}{a+b}$.

Hence to find the probability of an event, *divide the number of cases that favor it by the whole number of cases for and against it.*

For example, if in a lottery there are 5 prizes and 22 blanks, the probability that a person holding 1 ticket will win a prize is $\frac{5}{27}$, and the probability of his not winning is $\frac{22}{27}$.

EXAMPLE 1. From a bag containing 8 white, 7 black, and 5 red balls, one ball is drawn. Find the chance, (1) that it is white, (2) that it is black or red.

In all there are 20 ways of drawing a ball; of these 20 ways 8 are favorable to drawing a white ball, and 12 to drawing a black or a red ball; hence the chance of the ball being white is $\frac{8}{20}$, or $\frac{2}{5}$, and that of its being black or red is $\frac{3}{5}$.

EXAMPLE 2. From a bag containing 7 white and 4 red balls, 3 balls are drawn at random. Find the chance of these being all white.

The whole number of ways in which 3 balls can be drawn is ${}^{11}C_8$; and the number of ways of drawing 3 white balls is ${}^{7}C_8$; therefore, of drawing 3 white balls

$$\text{the chance} = \frac{{}^{7}C_8}{{}^{11}C_8} = \frac{7 \cdot 6 \cdot 5}{11 \cdot 10 \cdot 9} = \frac{7}{33}.$$

353. Unit of Probability. The sum of the probabilities $\dfrac{a}{a+b}$ and $\dfrac{b}{a+b}$ is unity; hence if the probability that an event will happen is p, the probability that it will fail is $1-p$.

If b is zero, the event is certain to happen, and its probability is unity; hence *certainty* is the *unit of probability.*

Instead of saying that the probability of an event is $\dfrac{a}{a+b}$, we sometimes say that *the odds are* a *to* b *in favor of the event, or* b *to* a *against it.*

EXAMPLE 1. Find the chance of throwing at least one ace in a single throw with three dice.

Here it is simpler to first find the chance of *not* throwing an ace. Each die can be thrown in five ways so as not to give an ace; hence the three can be thrown in 5^3, or 125, ways that will exclude aces (§ 339). The total number of ways of throwing 3 dice is 6^3, or 216. Hence the chance of *not* throwing one or more aces is $125 \div 216$; so that the chance of throwing at least one ace is $1-\frac{125}{216}$, or $\frac{91}{216}$ (§ 353).

Here the odds against the event are 125 to 91.

EXAMPLE 2. A has 3 shares in a lottery in which there are 4 prizes and 7 blanks; B has 1 share in a lottery in which there is 1 prize and 10 blanks; show that A's chance of success is to B's as 26 is to 3.

A can get all blanks in ${}^7C_3$, or 35, ways; he can draw 3 tickets in ${}^{11}C_3$, or 165, ways; hence A's chance of failure $=\frac{35}{165}=\frac{7}{33}$. Therefore A's chance of success $=1-\frac{7}{33}=\frac{26}{33}$.

B's chance of success is evidently $\frac{1}{11}$;

$$\therefore \text{ A's chance : B's chance} = \tfrac{26}{33} : \tfrac{1}{11} = 26 : 3.$$

Or to find A's chance we may reason thus: A may draw 3 prizes in $^4C_3$, or 4, ways; he may draw 2 prizes and 1 blank in $^4C_2 \times 7$, or 42, ways; he may draw 1 prize and 2 blanks in $4 \times {}^7C_2$, or 84, ways; hence A can succeed in $4 + 42 + 84$, or 130, ways. Therefore A's chance of success $= \frac{130}{165} = \frac{26}{33}$.

EXERCISE 46.

1. From the vessel on which Mr. A took passage one person has been lost overboard. There were 60 passengers and 30 in the crew. Find, (1) the chance that Mr. A is safe, (2) the chance that all the passengers are safe, (3) the probability that a passenger is lost.

2. There are 15 persons sitting around a table; find the probability that any 2 given persons sit together.

Wherever one of the 2 persons sits, the other may occupy any one of 14 places, of which 2 will put the 2 persons together.

3. According to the Carlisle Table of Mortality, it appears that out of 6335 persons living at the age of 14 years, only 6047 reach the age of 21 years. Find the probability that a child aged 14 years will reach the age of 21 years. Find the chance that he will not reach it.

4. From a bag containing 4 red and 6 black balls, 2 balls are drawn; find the chance, (1) that both are red, (2) that both are black, (3) that one is red and the other black.

5. From a bag containing 4 white, 5 black, and 6 red balls, 3 balls are drawn; find the probability that (1) all are white, (2) all black, (3) all red, (4) 2 black and 1 red, (5) 1 white and 2 black. Ans. $\frac{4}{455}$, $\frac{2}{91}$, $\frac{4}{91}$, $\frac{12}{91}$, $\frac{8}{91}$.

6. When two coins are thrown, find the chance that the result will be, (1) both heads, (2) both tails, (3) head and tail.

7. When two dice are thrown, what is the probability of throwing, (1) a 5 and 6, (2) two 6's?

8. From a committee of 7 Republicans and 6 Democrats, a sub-committee of 3 is chosen by lot. What is the probability that it will be composed of 2 Republicans and 1 Democrat?

9. From a committee of 8 Democrats, 7 Republicans, and 3 Independents, a sub-committee of 4 is chosen by lot. Find the chance that it will consist, (1) of 2 Democrats and 2 Republicans, (2) of 1 Democrat, 2 Republicans, and 1 Independent, (3) of 4 Democrats.

10. In a single throw with two dice, show that the chance of throwing 5 is $\frac{1}{9}$; of throwing 6 is $\frac{5}{36}$.

11. One of two events must happen, and the chance of the first is two thirds that of the second; find the odds in favor of the second.

12. In a bag are 4 white and 6 black balls; find the chance that, out of 5 drawn, 2, and 2 only, shall be white.

13. In Example 12 show that the chance of 2 *at least* being white is $\frac{31}{42}$.

14. Out of 100 mutineers, a general orders two men, chosen by lot, to be shot; the real leaders of the mutiny being 10, find the chance that, (1) one of the leaders will be taken, (2) two of them.

15. A has 3 shares in a lottery containing 3 prizes and 9 blanks; B has 2 shares in a lottery containing 2 prizes and 6 blanks; compare their chances of success.

16. There are 4 half-dollars and 3 quarter-dollars placed at random in a line; prove that the chance of the extreme coins being both quarter-dollars is $\frac{1}{7}$. In the case of m half-dollars and n quarter-dollars, show that the chance is

$$\frac{n(n-1)}{m(m-1)}.$$

17. There are three works, one consisting of 3 volumes, another of 4, and the third of 1 volume. They are placed on a shelf at random; prove that the odds against the volumes of the same works being all together is 137 to 3.

18. A man wants a particular span of horses from a stud of 8. His groom brings him 5 horses taken at random. What is the chance that both horses of the span are among them?

19. Of the three events A, B, C, one must, and only one can, occur; A can occur in a ways, B in b ways, and C in c ways, all the ways being equally likely; find the chance of each event.

Compound Events.

354. Thus far we have considered only single events. The concurrence of two or more events is sometimes called a **Compound** event.

Events are said to be *Dependent* or *Independent*, according as the happening (or failing) of one event *does* or *does not* affect the occurrence of the other.

355. ***The probability that two independent events will both happen is equal to the product of their separate probabilities.***

Suppose that the first event may happen in a ways and fail in b ways, all these cases being equally likely; and suppose that the second event may happen in a' ways and fail in b' ways, all these cases being equally likely. Each of the $a + b$ cases may be associated with each of the $a' + b'$ cases to form $(a + b)(a' + b')$ compound cases, all equally likely to occur.

In $a a'$ of these compound cases both events happen, in $b b'$ of them both fail, in $a b'$ of them the first happens and the second fails, and in $a' b$ of them the first fails and the second happens. Hence

$$\frac{a a'}{(a + b)(a' + b')} = \text{the chance that both events happen;}$$

$$\frac{b b'}{(a + b)(a' + b')} = \text{the chance that both events fail;}$$

$$\frac{a b'}{(a + b)(a' + b')} = \left\{ \begin{array}{l} \text{the chance that the first happens and} \\ \quad \text{the second fails;} \end{array} \right.$$

$$\frac{a' b}{(a + b)(a' + b')} = \left\{ \begin{array}{l} \text{the chance that the first fails and the} \\ \quad \text{second happens.} \end{array} \right.$$

356. ***The probability that any number of independent events will all happen is equal to the product of their separate probabilities.***

Let p_1, p_2, and p_3 be the respective probabilities of three independent events. The probability of the concurrence of the first and second events is $p_1 p_2$;

the probability of the concurrence of the first two events and the third is $(p_1 p_2)\, p_3$, or $p_1 p_2 p_3$; and so on for any number of events.

357. By § 356, if p is the chance that an event will happen in one trial, the chance of its happening each time in n trials is p^n.

358. The chance that all three of the events in § 356 will fail is $(1 - p_1)(1 - p_2)(1 - p_3)$.

Hence the chance that some one at least of them will happen is $1 - (1 - p_1)(1 - p_2)(1 - p_3)$.

The chance that the first two will happen and the third fail is $p_1 p_2 (1 - p_3)$.

EXAMPLE 1. Find the chance of throwing an ace in the first only of 2 successive throws with a single die.

The chance of throwing an ace in the first throw $= \frac{1}{6}$.

The chance of not throwing an ace in the second throw $= \frac{5}{6}$.

Hence the chance of the compound event $= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$.

EXAMPLE 2. From a bag containing 6 white and 9 black balls, 2 drawings are made, each of 3 balls, the balls first drawn being replaced before the second trial; find the chance that the first drawing will give 3 white, and the second 3 black balls.

The number of ways of drawing 3 balls $= {}^{15}C_3$;

" " " " 3 white $= {}^{6}C_3$;

" " " " 3 black $= {}^{9}C_3$.

Hence, of drawing 3 white balls at first trial

$$\text{the chance} = \frac{{}^{6}C_3}{{}^{15}C_3} = \frac{6 \cdot 5 \cdot 4}{15 \cdot 14 \cdot 13} = \frac{4}{91};$$

and, of drawing 3 black balls at second trial

$$\text{the chance} = \frac{{}^9C_3}{{}^{15}C_3} = \frac{9 \cdot 8 \cdot 7}{15 \cdot 14 \cdot 13} = \frac{12}{65}.$$

Hence the chance of the compound event $= \frac{4}{91} \times \frac{12}{65} = \frac{48}{5915}.$

EXAMPLE 3. If the odds are 11 to 9 against a person A, who is now 38 years old, living till he is 68, and 4 to 3 against a person B, now 43, living till he is 73; find the chance that one at least of these persons will be alive 30 years hence.

The chance that A will die within 30 years $= \frac{11}{20}$; the chance that B will die within 30 years $= \frac{4}{7}$; hence the chance that both will die $= \frac{11}{20} \times \frac{4}{7} = \frac{11}{35}$; therefore the chance that both will *not* die, that is, that one at least will be alive, $= 1 - \frac{11}{35} = \frac{24}{35}$.

EXAMPLE 4. In how many trials will the probability of throwing an ace with a single die amount to $\frac{2}{3}$?

Let $x =$ the number of trials. By § 357, the chance of failing to throw an ace each time in x trials is $(\frac{5}{6})^x$. Hence the chance of throwing an ace once at least in x trials is $1 - (\frac{5}{6})^x$;

$$\therefore \quad 1 - (\tfrac{5}{6})^x = \tfrac{2}{3}, \text{ or } (\tfrac{5}{6})^x = \tfrac{1}{3};$$

$$\therefore \quad x = \frac{\log 3}{\log 6 - \log 5} = \frac{0.4771213}{0.0791813} = 6.02.$$

Hence in 6 trials the chance of success is a little less than $\frac{2}{3}$, and in 7 trials it is greater than $\frac{2}{3}$.

359. Dependent Events. A slight modification of the meaning of a' and b' in § 355 enables us to estimate the chance of the concurrence of two *dependent* events. Thus, if after the first event has happened, a' denote the number of ways in which the second can follow, and b' the number of ways in which it will not follow; then the number of cases in which the two

events will both happen is aa', and the chance of their concurrence is $\dfrac{aa'}{(a+b)(a'+b')}$.

Hence if p is the chance of the first of two dependent events, and p' the chance that the second will follow, the chance of their concurrence is pp'.

EXAMPLE. From a bag containing 6 white and 9 black balls, two drawings are made, each of 3 balls, the balls first drawn *not* being replaced before the second trial; find the chance that the first drawing will be 3 white and the second 3 black balls.

At the first trial, 3 balls may be drawn in $^{15}C_3$ ways; and 3 white balls may be drawn in $^{6}C_3$ ways; hence the chance of 3 white balls at the first trial $= \dfrac{^{6}C_3}{^{15}C_3} = \dfrac{6 \cdot 5 \cdot 4}{15 \cdot 14 \cdot 13} = \dfrac{4}{91}$.

After 3 white balls have been drawn the bag contains 3 white and 9 black balls; therefore, at the second trial, 3 balls may be drawn in $^{12}C_3$ ways; and 3 black balls may be drawn in $^{9}C_3$ ways; hence, of drawing 3 black balls at the second trial,

$$\text{the chance} = \frac{^{9}C_3}{^{12}C_3} = \frac{9 \cdot 8 \cdot 7}{12 \cdot 11 \cdot 10} = \frac{21}{55}.$$

Hence the chance of the compound event $= \frac{4}{91} \times \frac{21}{55} = \frac{84}{5005}$.

The student should compare this result with that of Example 2 in § 358.

EXERCISE 47.

1. Show that the chance of throwing an ace in each of two successive throws with a single die is $\frac{1}{36}$.

2. Show that the chance of throwing an ace with a single die in two trials is $\frac{11}{36}$.

3. A traveller has 5 railroad connections to make in order to reach his destination on time. The chances are 3 to 1 in favor of each connection. What is the probability of his making them all?

4. Mr. A takes passage on a ship for London. The probability that the ship will encounter a gale is $\frac{1}{3}$. The probability that she will spring a leak in a gale is $\frac{1}{5}$. In case of a leak, the probability that the engines will be able to pump her out is $\frac{6}{7}$. If they fail, the probability that the compartments will keep her afloat is $\frac{3}{4}$. If she sinks, it is an even chance that any one passenger will be saved by the boats. What is the probability that Mr. A will be lost at sea on the voyage?

5. In how many trials will the chance of throwing an ace with a single die amount to $\frac{1}{2}$?

Ans. In 4 trials the chance is a little greater than $\frac{1}{2}$.

6. The odds against A's solving a certain problem are 1 to 2, and the odds in favor of B's solving the same problem are 3 to 4; find the chance that the problem will be solved if they both try.

7. The chance that a man will die within ten years is $\frac{1}{5}$, that his wife will die is $\frac{1}{8}$, and that his son will die is $\frac{1}{6}$; find the chance that at the end of ten years, (1) all will be living, (2) all will be dead, (3) one at least will be living, (4) husband living, but wife and son dead, (5) wife living, but husband and son dead, (6) husband and wife living, but son dead.

8. A bag contains 2 white balls and 4 black ones. Five persons, A, B, C, D, E, in alphabetical order each draw one ball and keep it. The first one who draws a white ball is to receive a prize. Show that their respective chances of winning are as 5 : 4 : 3 : 2 : 1.

A's chance of winning the prize is easily obtained.

That B may win, A must fail. Hence to find B's chance, we find, (1) the chance that A fails, (2) the chance that if A fails B will win. We then take the product of these chances.

That C may win, (1) A must fail, (2) B must fail, (3) C must draw a white ball. Hence C's chance of winning is the product of the chances of these three events; and so on.

9. A and B have one throw each of a coin. If A throws head, he is to receive a prize; if A fails and B throws head, he is to receive the prize. If A and B both fail, C receives the prize. Find the chance of each man winning the prize.

10. From a bag containing 5 white and 8 black balls two drawings are made, each of 3 balls, the balls not being replaced before the second drawing; find the chance that the first drawing will give 3 white and the second 3 black balls.

11. In three throws with a pair of dice, find the chance of throwing doublets at least once.

12. Find the chance of throwing 6 with a single die at least once in 5 trials.

13. The odds against a certain event are 6 to 3, and the odds in favor of another event independent of the former are 7 to 5; find the chance that one at least of the events will happen.

14. A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced; a second drawing is then made; find the chance that the first number drawn is even and the second odd.

* **360.** *If an event can happen in two or more different ways*, which are mutually exclusive, *the chance that it will happen is the sum of the chances of its happening in these different ways.*

When these different ways are all equally probable, the proposition is merely a repetition of the definition of probability. When they are not equally probable, the proposition is often regarded as self-evident from that definition. It may, however, be proved as follows:

Let $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ be respectively the chances of the happening of an event in two ways that are mutually exclusive. Then out of $b_1 b_2$ cases there are $a_1 b_2$ cases in which the event may happen the first way, and $a_2 b_1$ cases in which the event may happen the second way; and these ways are mutually exclusive. Therefore out of $b_1 b_2$ cases, $a_1 b_2 + a_2 b_1$ cases are favorable to the event; hence the chance that the event will happen in one of these two ways is

$$\frac{a_1 b_2 + a_2 b_1}{b_1 b_2}, \text{ or } \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

Similar reasoning will apply to any number of exclusive ways in which an event may happen.

Hence if an event can happen in n ways which are mutually exclusive, and if $p_1, p_2, p_3, \cdots, p_n$ are the probabilities that the event will happen in these different ways respectively, the probability that it will happen in some one of these ways is $p_1 + p_2 + p_3 + \cdots + p_n$.

EXAMPLE 1. Find the chance of throwing at least 8 in a single throw with two dice.

8 can be thrown in 5 ways, $\therefore$ the chance of throwing $8 = \frac{5}{36}$;
9 can be thrown in 4 ways, $\therefore$ the chance of throwing $9 = \frac{4}{36}$;
10 can be thrown in 3 ways, $\therefore$ the chance of throwing $10 = \frac{3}{36}$;
11 can be thrown in 2 ways, $\therefore$ the chance of throwing $11 = \frac{2}{36}$;
12 can be thrown in 1 way, $\therefore$ the chance of throwing $12 = \frac{1}{36}$.

These ways being mutually exclusive, and 8 at least being thrown in each case,

$$\text{the required chance} = \tfrac{5}{36} + \tfrac{4}{36} + \tfrac{3}{36} + \tfrac{2}{36} + \tfrac{1}{36} = \tfrac{5}{12}.$$

EXAMPLE 2. One purse contains 2 dollars and 4 half-dollars, a second 3 dollars and 5 half-dollars, a third 4 dollars and 2 half-dollars. If a coin is taken from one of these purses selected at random, find the chance that it is a dollar.

The chance of selecting the 1st purse $= \frac{1}{3}$;
the chance of then drawing a dollar $= \frac{2}{6} = \frac{1}{3}$;
$\therefore$ the chance of drawing a dollar from 1st purse $= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Similarly, the chance of drawing a dollar from 2d purse $= \frac{1}{8}$;
and the chance of drawing a dollar from 3d purse $= \frac{2}{9}$.

$$\therefore \text{The required chance} = \tfrac{1}{9} + \tfrac{1}{8} + \tfrac{2}{9} = \tfrac{11}{24}.$$

It is very important to note that when, as in the two examples given above, the probability of an event is the sum of the probabilities of two or more separate events, *these separate events must be mutually exclusive.*

*** 361.** *If* p *denote the chance of an event happening in one trial, and* $q = 1 - p$; *then the chance of its happening* r *times exactly in* n *trials is* $^nC_r\, p^r\, q^{n-r}$.

For if we select any particular set of r trials out of the whole number n, the chance that the event will happen in every one of these r trials and fail in the rest is $p^r q^{n-r}$ (§§ 355, 357); and as in the n trials there are ^nC_r sets of r trials, which are mutually exclusive and equally applicable, the chance that the event will happen r times exactly in n trials is

$$^nC_r p^r q^{n-r}.$$

*** 362.** *The chance that an event will happen at least* r *times in* n *trials is*

$$p^n + {^nC_1}\, p^{n-1} q + {^nC_2} p^{n-2} q^2 + \cdots + {^nC_{n-r}}\, p^r q^{n-r},$$

or the sum of the first $n - r + 1$ *terms of the expansion of* $(p + q)^n$.

For an event happens at least r times in n trials, if it happens n times, or $n - 1$ times, or $n - 2$ times, ..., or r times; and if in $^nC_r p^r q^{n-r}$ we put r equal to n, $n - 1$, $n - 2$, ..., r, in succession, and add the results, remembering that $^nC_{n-r} = {^nC_r}$, we obtain the expression given above.

EXAMPLE. In 5 throws with a single die, find, (1) the chance of throwing *exactly* 3 aces, (2) the chance of throwing *at least* 3 aces.

Here $p=\frac{1}{6}$, $q=\frac{5}{6}$, $n=5$, $r=3$; hence
the chance of throwing *exactly* 3 aces $= {}^5C_3\,(\frac{1}{6})^3\,(\frac{5}{6})^2=\frac{250}{7776}$.

The chance of throwing *at least* 3 aces is the sum of $n-r+1$, or 3, terms of the expansion of $(\frac{1}{6}+\frac{5}{6})^5$, or

the chance $=(\frac{1}{6})^5+5\,(\frac{1}{6})^4(\frac{5}{6})+10\,(\frac{1}{6})^3\,(\frac{5}{6})^2=\frac{276}{7776}$.

*363. **Expectation.** If p be a person's chance of winning a sum of money M, then Mp is called his *expectation*, or the value of his hope. The phrase *probable value* is often applied to things in the same way that expectation is to persons.

EXAMPLE. A and B take turns in throwing a die, and he who first throws a 6 wins a stake of \$22. If A throws first, find their respective expectations.

In his first throw, A's chance is $\frac{1}{6}$; in his second throw, it is $(\frac{5}{6})^2\times\frac{1}{6}$; in his third, it is $(\frac{5}{6})^4\times\frac{1}{6}$; and so on.

Hence A's chance $=\frac{1}{6}\,\{1+(\frac{5}{6})^2+(\frac{5}{6})^4+\dots\}$.

Similarly, B's chance $=\frac{5}{6}\cdot\frac{1}{6}\,\{1+(\frac{5}{6})^2+(\frac{5}{6})^4+\dots\}$.

Hence A's chance is to B's as 6 is to 5; or their respective chances are $\frac{6}{11}$ and $\frac{5}{11}$.

Therefore their expectations are \$12 and \$10 respectively.

NOTE. The theory of probability has its most important applications in *Insurance* and the calculation of *Probable Error* in physical investigations. It is also applied to testimony and causes. But the limits of this treatise exclude further consideration either of the theory or its applications. For a fuller treatment the student may consult Hall and Knight's Higher Algebra, Todhunter's Algebra, Whitworth's "Choice and Chance," and the articles Annuities, Insurance, and Probability in the "Encyclopædia Britannica." A complete account of the origin and development of the subject is given in Todhunter's "History of the Theory of Probability from the time of Pascal to that of Laplace."

EXERCISE 48.

1. Find the chance of throwing 9 at least in a single throw with two dice.

2. One compartment of a purse contains 3 half-dollars and 2 dollars, and the other 2 dollars and 1 half-dollar. A coin is taken out of the purse; show that the chance of its being a dollar is $\frac{8}{15}$.

3. If 8 coins are tossed, find the chance, (1) that there will be exactly 3 heads, (2) that there will be at least 3 heads.

4. If on an average 1 vessel in every 10 is wrecked, find the chance that out of 5 vessels expected, (1) exactly 4 will arrive safely, (2) 4 at least will arrive safely.

5. If 3 out of 5 business men fail, find the chance that out of 7 business men, (1) exactly 5 will fail, (2) 5 at least will fail.

6. Two persons, A and B, engage in a game in which A's skill is to B's as 3 to 4; find A's chance of winning at least 3 games out of 5.

7. If A's chance of winning a single game against B is $\frac{4}{7}$, find the chance, (1) of his winning exactly 3 games out of 4, (2) of his winning at least 3 games out of 4.

8. A person is allowed to draw two coins from a bag containing 4 dollars and 4 dimes; find the value of his expectation.

9. From a bag containing 6 dollars, 4 half-dollars, and 2 dimes, a person draws out 3 coins at random; find the value of his expectation.

10. Two persons toss a dollar alternately, on condition that the first who gets "heads" wins the dollar; find the expectation of each.

11. Find the worth of a lottery-ticket in a lottery of 100 tickets, having 4 prizes of $100, 10 of $50, and 20 of $5.

12. Three persons, A, B, and C, take turns in throwing a die, and he who first throws a 5 wins a prize of $182; show that their respective expectations are $72, $60, and $50.

13. A has 3 shares in a lottery in which there are 3 prizes and 6 blanks; B has 1 share in a lottery in which there is 1 prize and 2 blanks. Compare their chances of success.

14. Show that the chance of throwing more than 15 in one throw with three dice is $\frac{5}{108}$.

15. Compare the chances of throwing 4 with one die, 8 with two dice, and 12 with three dice.

16. There are three events A, B, C, one of which must, and only one can, happen. The odds are 8 to 3 against A, 5 to 2 against B; find the odds against C.

17. A and B throw with two dice; if A throws 9, find B's chance of throwing a higher number.

18. The letters in the word *Vermont* are placed at random in a row; find the chance that any two given letters, as the two vowels, are together.

CHAPTER XX.

CONTINUED FRACTIONS.

364. An expression of the general form

$$a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{f}{g+\cdots}}}$$

is called a **Continued Fraction.**

We shall consider only the simple form

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3}+\cdots},$$

in which a_1, a_2, a_3, $\cdots$ are positive integers.

This is often written in the more compact form

$$a_1+\frac{1}{a_2+}\;\frac{1}{a_3+}\;\frac{1}{a_4+}\cdots$$

The quantities a_1, a_2, a_3, $\cdots$ are called *quotients*, or *partial quotients*. A continued fraction is said to be *terminating* or *non-terminating*, according as the number of the quotients a_1, a_2, a_3, $\cdots$ is limited or unlimited. Any terminating continued fraction can evidently be reduced to an ordinary fraction by simplifying the fractions in succession, beginning from

the lowest. Hence any terminating continued fraction is a commensurable quantity.

$$\text{Thus,}\quad a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = a_1 + \frac{a_3}{a_2 a_3 + 1} = \frac{a_3(a_1 a_2 + 1) + a_1}{a_3 a_2 + 1}.$$

365. *To convert a given fraction into a continued fraction.*

Let $\frac{m}{n}$ be the given fraction; divide m by n, and let a_1 be the integral quotient and r_1 the remainder;

$$\text{then}\qquad \frac{m}{n} = a_1 + \frac{r_1}{n} = a_1 + \cfrac{1}{\cfrac{n}{r_1}}.$$

Divide n by r_1 with quotient a_2 and remainder r_2,

$$\text{then}\qquad \frac{n}{r_1} = a_2 + \frac{r_2}{r_1} = a_2 + \cfrac{1}{\cfrac{r_1}{r_2}}.$$

Divide r_1 by r_2 with quotient a_3 and remainder r_3; and so on.

$$\text{Hence}\quad \frac{m}{n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}},\quad \text{or}\quad a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots$$

If $m < n$, $a_1 = 0$, and a_2 is obtained by dividing n by m.

The above process is evidently the same as that of finding the G. C. D. of m and n; therefore if m and n are commensurable, we must at length obtain 0 as the remainder, and the process terminates.

Hence any fraction whose terms are commensurable can be converted into a *terminating* continued fraction.

EXAMPLE 1. Reduce $\frac{1051}{329}$ to a continued fraction.

Here the quotients are 3, 5, 7, 9;

$$\therefore \frac{1051}{329} = 3 + \frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9}.$$

EXAMPLE 2. Reduce $\frac{329}{1051}$ to a continued fraction.

Here the quotients are 0, 3, 5, 7, 9;

$$\therefore \frac{329}{1051} = \frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9}.$$

366. Convergents. The fractions obtained by stopping at the first, second, third, ..., quotients of a continued fraction, as

$$\frac{a_1}{1},\ a_1 + \frac{1}{a_2},\ a_1 + \frac{1}{a_2+}\,\frac{1}{a_3},\ \cdots$$

or when reduced to the common form,

$$\frac{a_1}{1},\ \frac{a_1 a_2 + 1}{a_2},\ \frac{a_3(a_1 a_2 + 1) + a_1}{a_3 a_2 + 1},\ \cdots$$

are called respectively the *first*, *second*, *third*, ..., *convergents*.

367. *The successive convergents are alternately less and greater than the continued fraction.*

The first convergent, a_1, is too small, since the part $\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$ is omitted. The second convergent,

$a_1 + \frac{1}{a_2}$, is too great, for the denominator a_2 is too small. The third convergent, $a_1 + \frac{1}{a_2 +}\frac{1}{a_3}$, is too small, for $a_2 + \frac{1}{a_3}$ is too great (a_3 being too small); and so on.

368. *To establish the law of formation of the successive convergents.*

If we consider the first three convergents,

$$\frac{a_1}{1}, \quad \frac{a_1 a_2 + 1}{a_2}, \quad \frac{a_3 (a_1 a_2 + 1) + a_1}{a_3 a_2 + 1}, \qquad \S\ 366.$$

we see that the numerator of the third convergent may be obtained by multiplying the numerator of the second convergent by the third quotient, and adding the numerator of the first convergent; also that the denominator may be formed in a similar manner from the denominators of the first two convergents. We proceed to show that this law holds for all subsequent convergents.

Let the numerators of the successive convergents be denoted by $p_1, p_2, p_3, \cdots$, and the denominators by $q_1, q_2, q_3, \cdots$ Assume that the law holds for the nth convergent;

then $$p_n = a_n p_{n-1} + p_{n-2}, \qquad (1)$$

and $$q_n = a_n q_{n-1} + q_{n-2}. \qquad (2)$$

The $(n+1)$th convergent evidently differs from the nth only in having $a_n + \dfrac{1}{a_{n+1}}$ in the place of a_n; hence

$$\frac{p_{n+1}}{q_{n+1}} = \frac{\left(a_n + \dfrac{1}{a_{n+1}}\right)p_{n-1} + p_{n-2}}{\left(a_n + \dfrac{1}{a_{n+1}}\right)q_{n-1} + q_{n-2}}$$

$$= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}}$$

$$= \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}, \text{ by (1) and (2).}$$

Hence the law holds for the $(n+1)$th convergent, if it holds for the nth. But it does hold for the third; hence it holds for the fourth; and so on.

Therefore it holds universally after the second.

The method of proof employed in this article is known as **Mathematical Induction.**

EXAMPLE 1. Calculate the successive convergents of

$$2 + \frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{11+}\,\frac{1}{2}.$$

Here $\quad a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,$

are $\quad 2,\ 6,\ 1,\ 1,\ 11,\ 2.$

Hence $\dfrac{p_1}{q_1} = \dfrac{2}{1}$; $\qquad \dfrac{p_2}{q_2} = 2 + \dfrac{1}{6} = \dfrac{13}{6}$;

$$\frac{p_3}{q_3} = \frac{1 \times 13 + 2}{1 \times 6 + 1} = \frac{15}{7}; \qquad \frac{p_4}{q_4} = \frac{1 \times 15 + 13}{1 \times 7 + 6} = \frac{28}{13};$$

$$\frac{p_5}{q_5} = \frac{11 \times 28 + 15}{11 \times 13 + 7} = \frac{323}{150}; \quad \frac{p_6}{q_6} = \frac{2 \times 323 + 28}{2 \times 150 + 13} = \frac{674}{313}.$$

EXAMPLE 2. Find the successive convergents of

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{6}.$$

Here $\quad a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,$

are $\quad 0,\ 2,\ 2,\ 3,\ 1,\ 4,\ 2,\ 6.$

Hence $\quad \frac{0}{1},\ \frac{1}{2},\ \frac{2}{5},\ \frac{7}{17},\ \frac{9}{22},\ \frac{43}{105},\ \frac{95}{232},\ \frac{613}{1497},$

are the successive convergents.

EXERCISE 49.

Compute the successive convergents to

1. $3+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3}.$

2. $\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{6+}\,\frac{1}{8+}\,\frac{1}{10}.$

3. $3+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9}.$

Express each of the following fractions as a continued fraction, and find its convergents:

4. $\frac{43}{30}$. 6. $\frac{406}{891}$. 8. $\frac{2991}{568}$. 10. $\frac{1189}{3927}$.

5. $\frac{67}{88}$. 7. $\frac{1709}{1380}$. 9. $\frac{833}{159}$. 11. $\frac{729}{2318}$.

Reduce to a continued fraction, and find the fourth convergent to, each of the following numbers:

12. 0.37. 13. 1.139. 14. 0.3029. 15. 4.316.

Write 0.37 as a common fraction, $\frac{87}{100}$; then proceed as above.

369. *The difference between any two consecutive convergents is unity divided by the product of their denominators; that is,*

$$\frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}}^* = \frac{1}{q_n q_{n+1}}, \text{ or } p_n q_{n+1} \sim p_{n+1} q_n = 1.$$

The law holds for the first two convergents, since

$$\frac{a_1}{1} \sim \frac{a_1 a_2 + 1}{a_2} = \frac{1}{a_2}. \qquad (1)$$

Assume that the law holds for $\frac{p_{n-1}}{q^{n-1}}$ and $\frac{p_n}{q_n}$, so that

$$p_n q_{n-1} \sim q_n p_{n-1} = 1; \qquad (2)$$

then by § 368

$$\begin{aligned} p_n q_{n+1} \sim p_{n+1} q_n &= p_n (a_{n+1} q_n + q_{n-1}) \sim q_n (a_{n+1} p_n + p_{n-1}) \\ &= p_n q_{n-1} \sim q_n p_{n-1} \\ &= 1, \text{ by } (2). \end{aligned}$$

Hence, if the law holds for one pair of consecutive convergents, it holds for the next pair. But by (1), the law does hold for the *first* pair; therefore it holds for the *second* pair; and so on.

Therefore it is universally true.

370. Any convergent $p_n \div q_n$ is in its lowest terms. For if p_n and q_n had a common factor, it would also be a factor of $p_n q_{n+1} \sim p_{n+1} q_n$, or unity; which is impossible.

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* The expression $x \sim y$ denotes "the difference between x and y."

371. Let x denote the value of any continued fraction; then, by § 367, x lies between any two consecutive convergents; hence

$$x \sim \frac{p_n}{q_n} < \frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}};$$

$$\therefore \quad x \sim \frac{p_n}{q_n} < \frac{1}{q_n q_{n+1}} < \frac{1}{(q_n)^2 a_{n+1}}. \quad \text{§§ 369, 368.}$$

That is, $\frac{p_n}{q_n}$ differs from x by less than $\frac{1}{(q_n)^2 a_{n+1}}$.

Hence the nth convergent is a near approximation when q_n and a_{n+1} are large.

372. *Each convergent is nearer to the continued fraction than any of the preceding convergents.*

Let x denote the continued fraction, and A the *complete* $(n+2)$th quotient, $a_{n+2} + \frac{1}{a_{n+3}} + \cdots$; then x differs from $\frac{p_{n+2}}{q_{n+2}}$ only in having A in the place of a_{n+2}; hence

$$x = \frac{A\,p_{n+1} + p_n}{A\,q_{n+1} + q_n}.$$

$$\therefore \; x \sim \frac{p_n}{q_n} = \frac{A\,(p_{n+1} q_n \sim p_n q_{n+1})}{q_n\,(A\,q_{n+1} + q_n)} = \frac{A}{q_n\,(A\,q_{n+1} + q_n)},$$

and

$$\frac{p_{n+1}}{q_{n+1}} \sim x = \frac{p_{n+1} q_n \sim p_n q_{n+1}}{q_{n+1}(A\,q_{n+1} + q_n)} = \frac{1}{q_{n+1}\,(A\,q_{n+1} + q_n)}.$$

Now $A > 1$, and $q_n < q_{n+1}$; hence the difference between the $(n+1)$th convergent and x is less than

the difference between the nth convergent and x; that is, any convergent is nearer to the continued fraction than the next preceding convergent, and therefore than any preceding convergent.

From this property and that of § 367, it follows that

The convergents of an odd order continually increase, but are always less than the continued fraction.

The convergents of an even order continually decrease, but are always greater than the continued fraction.

EXAMPLE. Find the successive convergents to 3.14159.

Here the quotients are 3, 7, 15, 1, 25, 1, 7, $\cdots$; hence the convergents are $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\cdots$

If the 4th convergent, which is greater than 3.14159, be taken as its value, the error will be less than $1 \div 25\,(113)^2$, and therefore less than $1 \div 25\,(100)^2$, or 0.000004.

The convergents above will be recognized as the approximate values of π, or the ratio of the circumference of a circle to its diameter. This example illustrates the use of the properties of continued fractions in approximating to the values of incommensurable ratios or those represented by large numbers.

373. *Any convergent approaches more nearly the value of the continued fraction,* x, *than any other fraction whose denominator is less than that of the convergent.*

For let the fraction $\frac{r}{s}$ be nearer to x than $\frac{p_n}{q_n}$; then is it nearer to x than the $(n-1)$th convergent (§ 372); and since x lies between the nth and the

$(n-1)$th convergent, $r \div s$ does also; hence we have

$$\frac{r}{s} \sim \frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}}, \text{ or } \frac{1}{q_n q_{n-1}}. \qquad \S\ 369$$

$$\therefore \quad r q_{n-1} \sim s p_{n-1} < \frac{s}{q_n}. \qquad (1)$$

Now the first member of (1) is an integer; hence $s > q_n$; that is, if $\frac{r}{s}$ is nearer x than is $\frac{p_n}{q_n}$, $s > q_n$.

374. Periodic Continued Fractions. A continued fraction in which the quotients recur is called a *periodic*, or *recurring*, *continued fraction*.

Any quadratic surd can be expressed as a periodic continued fraction. We give the following example to illustrate this principle, and to exhibit the use of the properties of continued fractions in approximating to the value of a quadratic surd.

EXAMPLE. Convert $\sqrt{15}$ into a continued fraction, and find its convergents.

Since 3 is the greatest integer in $\sqrt{15}$, we write

$$\sqrt{15} = 3 + \frac{\sqrt{15}-3}{1} = 3 + \frac{6}{\sqrt{15}+3}; \qquad (1)$$

$$\frac{\sqrt{15}+3}{6} = 1 + \frac{\sqrt{15}-3}{6} = 1 + \frac{1}{\sqrt{15}+3}; \qquad (2)$$

$$\frac{\sqrt{15}+3}{1} = 6 + \frac{\sqrt{15}-3}{1} = 6 + \frac{6}{\sqrt{15}+3}. \qquad (3)$$

The last fraction in (3) is the same as that in (1); hence after this the quotients 1, 6, will recur.

Hence $$\sqrt{15} = 3 + \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$$

The quotient in each of the identities, (1), (2), (3), is the greatest integer in the value of its first member. The numerator of each fraction is rationalized so that the *inverted* fraction will have a rational denominator.

To find the convergents, we have the quotients

$$3,\quad 1,\quad 6,\quad 1,\quad 6,\quad 1,\quad 6,\ \ldots;$$

$$\therefore\quad \tfrac{3}{1},\ \tfrac{4}{1},\ \tfrac{27}{7},\ \tfrac{31}{8},\ \tfrac{213}{55},\ \tfrac{244}{63},\ \ldots$$

are the first six convergents.

The error in taking the sixth convergent as the value of $\sqrt{15}$ is less than $1 \div 6\,(63)^2$, and therefore less than 0.00005.

375. Every periodic continued fraction is equal to one of the surd roots of a quadratic equation with rational coefficients. The following example will illustrate this general truth.

EXAMPLE. Express $1 + \dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\cdots$ as a surd.

Let x denote the value of the continued fraction; then

$$x - 1 = \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots$$

$$= \frac{1}{2+}\,\frac{1}{3+(x-1)};$$

$$\therefore\quad 2x^2 + 2x - 7 = 0.$$

The continued fraction, being positive, is equal to the positive root of this equation, or $\frac{1}{2}(\sqrt{15} - 1)$.

EXERCISE 50.

Reduce to a continued fraction, and find the sixth convergent to, each of the following surds:

1. $\sqrt{5}$. 3. $\sqrt{6}$. 5. $\sqrt{14}$. 7. $3\sqrt{5}$.

2. $\sqrt{8}$. 4. $\sqrt{13}$. 6. $\sqrt{22}$. 8. $4\sqrt{10}$.

9. In each of the above examples the difference between the surd and the sixth convergent is less than what?

10. Find the first convergent to

$$1+\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9+}\,\frac{1}{11+}\cdots$$

which differs from it by less than 0.0001.

11. Find the first convergent to $\sqrt{101}$ that differs from it by less than 0.0000004.

12. Given that a metre is equal to 1.0936 yards, show that the error in taking 222 yards as equivalent to 203 metres will be less than 0.000005.

13. A kilometre is very nearly equal to 0.62138 miles; show that the error in taking 103 kilometres as equivalent to 64 miles will be less than 0.000025.

14. Find the first six convergents to the ratio of a diagonal to a side of a square. The difference between each of the six convergents and the true ratio is less than what?

15. Express $3+\frac{1}{6+}\,\frac{1}{6+}\,\frac{1}{6+}\cdots$ as a surd.

16. Express $\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\cdots$ as a surd.

CHAPTER XXI.

THEORY OF EQUATIONS.

376. The General Equation. Let n be any positive integer, and $p_1, p_2, p_3, \cdots, p_n$, be any rational known quantities; then the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0 \quad \text{(A)}$$

will be the general type of a rational integral equation of the nth degree. In this chapter we shall let

$$F(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n$$

and write equation (A) briefly $F(x) = 0$.*

377. A Root of the equation $F(x) = 0$ is any value of x, real or imaginary, that causes the function, $F(x)$, to vanish.

378. Reduction to the form $F(x) = 0$. In general, any equation in x having rational coefficients can be transformed into an equation of the form $F(x) = 0$. The following example will illustrate the general truth.

* What properties of the equation $F(x) = 0$ belong also to the equation formed by putting any rational integral function of x equal to zero, the reader will readily discover.

EXAMPLE. Reduce $\dfrac{1 - x^{\frac{2}{3}}}{1 + x} = \dfrac{x^{-1} + 3}{x^{\frac{1}{2}} + 2}$ to the form of $F(x) = 0$.

Clearing the given equation of fractions we obtain

$$x^{\frac{1}{2}} - x^{\frac{7}{6}} + 2 - 2x^{\frac{2}{3}} = x^{-1} + 3 + 1 + 3x.$$

Multiplying by x to free of negative exponents, we obtain

$$x^{\frac{3}{2}} - x^{\frac{13}{6}} - 2x^{\frac{5}{3}} = 1 + 2x + 3x^2. \qquad (1)$$

To transform (1) into another equation with integral exponents, put $x = y^6$, 6 being the L. C. M. of the denominators of the fractional exponents of x. We thus obtain

$$y^9 - y^{13} - 2y^{10} = 1 + 2y^6 + 3y^{12},$$

or

$$y^{13} + 3y^{12} + 2y^{10} - y^9 + 2y^6 + 1 = 0, \qquad (2)$$

which is in the required form.

The roots of (1) and (2) hold the relation $x = y^6$.

EXERCISE 51.

Reduce the following equations to the form $F(x) = 0$:

1. $\dfrac{2}{x} - 3x + \frac{4}{5}x^{\frac{1}{2}} - 1 = 1.$

2. $\dfrac{x^2 - 1}{1 + x^{\frac{1}{2}}} = 1 - x^{-2}.$

3. $\sqrt{1 - x^2} = 1 - 3x^{\frac{1}{3}}.$

4. $\sqrt{2x - 3x^3} - x = \sqrt{1 - x}.$

379. Divisibility of $F(x)$. *If* F (x) *is divided by* x − a, *the remainder will be* F (a).

Divide $F(x)$ by $x - a$ until a remainder is obtained that does not involve x. Let $F_1(x)$ denote the quotient, and R the remainder; then

$$F(x) \equiv (x - a) F_1(x) + R.$$

Since R does not involve x, it is the same for all values of x. Putting $x = a$, we obtain

$$F(a) \equiv 0 \times F_1(a) + R \equiv R, \text{ the remainder.}$$

If $F(a) \equiv 0$, $F(x)$ *is divisible by* $x - a$; *and conversely.*

EXAMPLE. If n is even, show that $x^n - b^n$ is divisible by $x + b$.

Since n is even, and $F(x) \quad x^n - b^n$;

$$\therefore F(-b) = b^n - b^n \equiv 0.$$

Hence $x^n - b^n$ is divisible by $x - (-b)$, or $x + b$.

380. *If* a *is a root of the equation* $F(x) = 0$, *that is, if* $F(a) \equiv 0$, *then* $F(x)$ *is divisible by* $x - a$ (§ 379). Conversely, *if* $F(x)$ *is divisible by* $x - a$, *then* $F(a) \equiv 0$, *that is,* a *is a root of the equation* $F(x) = 0$.

381. Horner's Method of Synthetic Division.

Let it be required to divide

$$A x^3 + B x^2 + C x + D \text{ by } x - a.$$

In the usual method given below, for convenience we write the divisor to the right of the dividend and the quotient below it.

$$
\begin{array}{llll|l}
A x^3 + B x^2 & & + C x & + D & x^{*} - a \\
{}^{*}A x^3 - A a x^2 & & & & \hline {}^{*}A x^2 + {}^{*}(A a + B) x \\
\cline{1-1}
\quad (A a + B) x^2 & & & & + {}^{*}(A a^2 + B a + C) \\
\quad {}^{*}(A a + B) x^2 & - (A a^2 + B a) x & & & \\
\cline{2-2}
& (A a^2 + B a + C) x & & & \\
& {}^{*}(A a^2 + B a + C) x & - (A a^3 + B a^2 + C a) & & \\
\cline{3-4}
& & A a^3 + B a^2 + C a + D & &
\end{array}
$$

Here the remainder, $A a^3 + B a^2 + C a + D$, is the value of the dividend, $A x^3 + B x^2 + C x + D$, for $x = a$, which affords a second proof of § 379.

In the shorter or synthetic method, we write the coefficients of the dividend with a at their right as below:

$$
\begin{array}{llll|l}
A & + B & + C & + D & a \\
\cline{5-5}
& + A a & + A a^2 + B a & + A a^3 + B a^2 + C a & \\
\hline
& A a + B & A a^2 + B a + C & A a^3 + B a^2 + C a + D &
\end{array}
$$

Multiplying A by a, writing the product under B, and adding, we obtain $A a + B$. Multiplying this sum by a, writing the product under C, and adding, we obtain $A a^2 + B a + C$. In like manner the last sum is obtained.

Now A and the first two sums are respectively the coefficients of x^2, x, and x^0 in the quotient obtained above by the ordinary method, and the last sum is the remainder.

In like manner any rational integral function of x may be divided by $x - a$. If any power of x is missing, its coefficient is zero, and must be written in its place with the others.

The shorter or synthetic method of division is obtained from the usual method given above by omitting the powers of x and the terms marked with an asterisk (*), by changing the minus signs to plus (which is the same thing as changing the sign of the

second term of the divisor), and then adding instead of subtracting.

EXAMPLE 1. Divide $x^4 + x^3 - 29x^2 - 9x + 180$ by $x - 6$. Write the coefficients with 6 at their right and proceed as below.

$$\begin{array}{rrrrr|l} 1 & +1 & -29 & -9 & +180 & \underline{6} \\ & +6 & +42 & +78 & +414 & \\ \hline 1 & +7 & +13 & +69 & +594 & \end{array}$$

Thus the quotient $\equiv x^3 + 7x^2 + 13x + 69$,
and the remainder $\equiv F(6) \equiv 594$.

EXAMPLE 2. Divide $x^4 + x^3 - 29x^2 - 9x + 180$ by $x + 6$, or $x - (-6)$.

$$\begin{array}{rrrrr|l} 1 & +1 & -29 & -9 & +180 & \underline{-6} \\ & -6 & +30 & -6 & +90 & \\ \hline 1 & -5 & +1 & -15 & +270 & \end{array}$$

Here the quotient $\equiv x^3 - 5x^2 + x - 15$,
and the remainder $\equiv F(-6) \equiv 270$.

EXAMPLE 3. Divide $x^3 + 21x + 342$ by $x + 6$.

$$\begin{array}{rrrr|l} 1 & +0 & +21 & +342 & \underline{-6} \\ & -6 & +36 & -342 & \\ \hline 1 & -6 & +57 & 0 & \end{array}$$

Here the quotient $\equiv x^2 - 6x + 57$,
and the remainder $\equiv F(-6) \equiv 0$.
Hence the division is exact, and -6 is a root of $F(x) = 0$.

382. When one root of an equation is known, the equation may be *depressed* into another of the next lower degree, the roots of which are the remaining roots of the given equation.

EXAMPLE. Solve $x^3 - 12x^2 + 45x - 50 = 0$, one root being 5.

One root being 5, one factor of $F(x)$ is $x - 5$ (§ 380). By division the other factor of $F(x)$ is found to be $x^2 - 7x + 10$. Hence by § 137 the two roots required are those of the quadratic equation

$$x^2 - 7x + 10 = 0. \tag{1}$$

The roots of (1) are evidently 5 and 2.

EXERCISE 52.

By § 379, show that

1. When n is integral, $x^n - a^n$ is divisible by $x - a$.

2. When n is odd, $x^n + a^n$ is divisible by $x + a$.

3. When n is even, $x^n + a^n$ is *not* divisible by either $x - a$ or $x + a$.

By Horner's method divide

4. $x^3 - 2x^2 - 4x + 8$ by $x - 3$; by $x - 2$.

5. $2x^4 + 4x^3 - x^2 - 16x - 12$ by $x + 4$; by $x + 3$.

6. $3x^4 - 27x^2 + 14x + 120$ by $x - 6$; by $x + 5$.

7. Evaluate $2x^4 - 3x^3 + 3x - 1$ for $x = 4$, $x = -3$, $x = 3$.

8. One root of $x^3 - 6x^2 + 10x - 8 = 0$ is 4; find the others.

9. One root of $x^3 + 8x^2 + 20x + 16 = 0$ is -2; find the others.

10. One root of $x^3 + 2x^2 - 23x - 60 = 0$ is -3; find the others.

11. One root of $x^3 - 7x^2 + 36 = 0$ is -2; find the others.

12. Two roots of $x^4 + x^3 - 29x^2 - 9x + 180 = 0$ are 3 and -3; find the others.

13. Two roots of $x^4 - 4x^3 - 8x + 32 = 0$ are 2 and 4; find the others.

14. If the coefficients of $F(x)$ are all positive, the equation $F(x) = 0$ can have no positive root.

15. If the sum of the coefficients of the even powers of x in $F(x)$ is equal to the sum of those of the odd powers, one root of the equation $F(x) = 0$ is -1.

383. *Every equation of the form* F (x) = 0 *has a root, real or imaginary.*

For the proof of this theorem see Burnside and Panton's or Todhunter's "Theory of Equations." The proof is too long and difficult to be given here.

384. Number of Roots. *Every equation of the* n*th degree has* n, *and only* n, *roots.*

By § 383, the equation $F(x) = 0$ has a root. Let a_1 denote this root; then, by § 380, $F(x)$ is divisible by $x - a_1$, so that

$$F(x) \equiv (x - a_1) F_1(x), \qquad (1)$$

in which $F_1(x)$ has the form of $F(x)$, and is of the $(n-1)$th degree. Now the equation $F_1(x)=0$ has a root. Denote this root by a_2; then

$$F_1(x) \equiv (x-a_2)\,F_2(x), \qquad (2)$$

in which $F_2(x)$ is of the $(n-2)$th degree.

Repeating this process n times we finally obtain

$$F_{n-1}(x) = x - a_n. \qquad \text{(n)}$$

From (1), (2), ..., (n), we obtain

$$\begin{aligned} F(x) &\equiv (x-a_1)\,F_1(x) \\ &\equiv (x-a_1)(x-a_2)\,F_2(x) \\ &\quad \cdots \qquad \cdots \qquad \cdots \\ &\equiv (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n). \qquad (3) \end{aligned}$$

Now, since $F(x)$ vanishes when x has any one of the values $a_1, a_2, a_3, \ldots, a_n$, and only then, the equation $F(x)=0$ has n, and only n, roots.

385. Equal Roots. If two or more of the factors $x-a_1$, $x-a_2$, ..., $x-a_n$ are equal, the equation $F(x)=0$ is considered as having two or more *equal* roots.

Thus, of the equation $(x-4)^3(x+5)^2(x-7)=0$, three roots are 4 each, and two are -5 each.

386. Equation having Given Roots. An equation can be formed of which the roots shall be any given

quantities, by subtracting each of these quantities from x, and putting the product of the results equal to zero.

Thus, the equation whose roots are 2, 3, and -1 is

$$(x-2)(x-3)(x+1)=0, \quad \text{or} \quad x^3-4x^2+x+6=0.$$

387. Relations between Coefficients and Roots.

In the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\cdots+p_{n-1}x+p_n=0 \quad \text{(A)}$$

$-\mathrm{p}_1 =$ *the sum of the roots;*

$\mathrm{p}_2 =$ *the sum of the products of the roots taken two at a time;*

$-\mathrm{p}_3 =$ *the sum of the products of the roots taken three at a time.*

... ...

$(-1)^n \mathrm{p}_n =$ *the product of the* n *roots.*

If $a_1, a_2, a_3, \cdots, a_n$ denote the n roots of (A), by (3) of § 384 we have the identity

$$x^n+p_1x^{n-1}+\cdots+p_n \equiv (x-a_1)(x-a_2)\cdots(x-a_n).$$

Multiplying together the factors of the second member, and equating the coefficients of like powers of x (§ 263), we obtain the theorem.

Thus, when $x = 2$, we have

$$x^2+p_1x+p_2 \equiv x^2-(a_1+a_2)x+a_1a_2;$$

$$\therefore \; -p_1 = a_1+a_2, \; p_2 = a_1a_2.$$

When $n = 3$, we have

$$x^3 + p_1 x^2 + p_2 x + p_3$$
$$\equiv x^3 - (a_1 + a_2 + a_3) x^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3) x - a_1 a_2 a_3;$$
$$\therefore \quad -p_1 = a_1 + a_2 + a_3, \quad p_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, \quad -p_3 = a_1 a_2 a_3.$$

From the laws of multiplication it is evident that the same relation holds when $n = 4, 5, 6, \cdots$

If the term in x^{n-1} is wanting, the sum of the roots is 0, and if the known term is wanting, at least one root is 0.

Thus, in the equation $x^4 + 6x^2 - 11x - 6 = 0$, the sum of the roots is 0; the sum of their products taken two at a time is 6; the sum of their products taken three at a time is 11; and their product is -6. Note that $-p_1 = (-1)p_1$, $p_2 = (-1)^2 p_2, \cdots$

NOTE. The coefficients in any equation are functions of the roots; and conversely, the roots are functions of the coefficients. The roots of a literal quadratic equation have been expressed in terms of the coefficients (§ 144). The roots of a literal cubic or biquadratic equation may also be expressed in terms of the coefficients, as will be shown in §§ 421, 423. But the roots of a *literal* equation of the fifth or higher degree cannot be so expressed, as was proved by Abel in 1825.

EXERCISE 53.

By § 386, form the equations whose roots are given below, and verify each equation by § 387:

1. $1, -3, -5$.
2. $1, \sqrt{2}, -\sqrt{2}$.
3. $1, \pm\sqrt{3}, \pm\sqrt{5}$.
4. $-\frac{2}{3}, 3 \pm \sqrt{-7}, 5$.
5. $1, 2, \sqrt{3}$.
6. $3, -4, \sqrt{-2}$.

7. $\frac{3}{4}, 1 \pm \sqrt{3}, 1 \pm \sqrt{5}$.

9. $\sqrt{3}, \sqrt{-2}$.

8. $\pm \sqrt{-1}, 3 \pm \sqrt{-2}, 2$.

NOTE. In each of the above examples, the student should note that the coefficients of the equation obtained are *all* rational whenever the surd or imaginary roots occur in conjugate pairs.

10. Solve $4x^3 - 24x^2 + 23x + 18 = 0$, having given that its roots are in arithmetical progression.

Reduce the equation to the form $F(x) = 0$, and denote its roots by $a - b$, a, and $a + b$; then by § 387

$$3a = 6,\ 3a^2 - b^2 = \tfrac{23}{4},\ a(a^2 - b^2) = -\tfrac{9}{2}. \qquad (1)$$

Hence $a = 2$, and $b = \pm \frac{5}{2}$; therefore the roots are $-\frac{1}{2}$, 2, $\frac{9}{2}$. The values of a and b must satisfy all three of the equations in (1).

11. Solve $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.

12. Solve $4x^3 + 20x^2 - 23x + 6 = 0$, two of the roots being equal.

13. Solve $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in geometrical progression.

388. Imaginary Roots. *In the equation* $\mathrm{F}(\mathrm{x}) = 0$, *imaginary roots occur in conjugate pairs; that is, if* $\mathrm{a} + \mathrm{b}\sqrt{-1}$ *is a root of* $\mathrm{F}(\mathrm{x}) = 0$, *then* $\mathrm{a} - \mathrm{b}\sqrt{-1}$ *is also a root.*

If $a + b\sqrt{-1}$ be substituted for x in $F(x)$, all its terms will be real except those containing odd powers of $b\sqrt{-1}$, which will be imaginary. Representing the sum of all the real terms by A, and the sum of all the imaginary terms by $B\sqrt{-1}$, we have

$$F(a + b\sqrt{-1}) \equiv A + B\sqrt{-1} \equiv 0. \qquad (1)$$

Now $F(a - b\sqrt{-1})$ will evidently differ from $F(a + b\sqrt{-1})$ only in the signs of the terms containing the odd powers of $b\sqrt{-1}$; that is, in the sign of $B\sqrt{-1}$; hence

$$F(a - b\sqrt{-1}) \equiv A - B\sqrt{-1}.$$

From (1) by § 114, $A = 0$, and $B = 0$; hence

$$F(a - b\sqrt{-1}) \equiv 0.$$

EXAMPLE. One root of $x^3 - 4x^2 + 4x - 3 = 0$ is $\frac{1}{2}(1 + \sqrt{-3})$; find the others.

Since $\frac{1}{2}(1 + \sqrt{-3})$ is one root, $\frac{1}{2}(1 - \sqrt{-3})$ is a second root. The sum of these two roots is 1, and by § 387 the sum of all three roots is 4; hence the third root must be 3.

$$\text{Here } F(x) \equiv (x - 3)\,(x - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3})\,(x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3})$$
$$\equiv (x - 3)[(x - \tfrac{1}{2})^2 + \tfrac{3}{4}] \equiv (x - 3)(x^2 - x + 1);$$

that is, the *real* factors of $x^3 - 4x^2 + 4x - 3$ are $x - 3$ and $x^2 - x + 1$.

389. By § 388 an equation of an odd degree must have at least one real root; while an equation of an even degree may not have any real root.

390. Real Factors of $F(x)$. Since $(x-a-b\sqrt{-1})(x-a+b\sqrt{-1}) \equiv (x-a)^2+b^2$, the imaginary factors of $F(x)$ occur in conjugate pairs whose products are of the form $(x-a)^2+b^2$.

Hence, F(x) *can be resolved into real linear or quadratic factors in* x.

391. *To transform an equation into another whose roots shall be some multiple of those of the first.*

If in the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\cdots+p_n=0, \quad (1)$$

we put $x=x_1 \div a$, and multiply by a^n, we obtain

$$x_1^n+p_1ax_1^{n-1}+p_2a^2x_1^{n-2}+p_3a^3x_1^{n-3}+\cdots+p_na^n=0. \quad (2)$$

Since $x_1=ax$, the roots of (2) are a times those of (1).

Hence, to effect the required transformation, *multiply the second term of* F(x) = 0 *by the given factor, the third by its square, and so on.*

Any missing power of x must be written with zero as its coefficient before the rule is applied.

The chief use of this transformation is to clear an equation of fractional coefficients.

EXAMPLE. Transform the equation $x^3-\frac{5}{2}x^2+\frac{7}{4}x-\frac{3}{16}=0$ into another with integral coefficients.

Multiplying the second term by a, the third by a^2, the fourth by a^3, we obtain

$$x^3 - \tfrac{5}{2} a x^2 + \tfrac{7}{4} a^2 x - \tfrac{3}{16} a^3 = 0. \qquad (1)$$

By inspection we discover that 4 is the least value of a that will render the coefficients of (1) integral. Putting $a = 4$, we have

$$x^3 - 10 x^2 + 28 x - 12 = 0 \qquad (2)$$

as the equation required.

The roots of (2) each divided by 4 are the roots of (1).

EXERCISE 54.

1. One root of $x^3 - 6x^2 + 57x - 196 = 0$ is $1 - 4\sqrt{3}$; find the others.

2. One root of $x^3 - 6x + 9 = 0$ is $\frac{1}{2}(3 + \sqrt{-3})$; find the others.

3. Two roots of $x^6 - x^5 + x^4 - x^2 + x - 1 = 0$ are $-\sqrt{-1}$ and $\frac{1}{2}(1 + \sqrt{-3})$; find the others.

4. One root of $x^3 - 2x^2 + 2x - 1 = 0$ is $\frac{1}{2}(1 + \sqrt{-3})$; find the real factors of $x^3 - 2x^2 + 2x - 1$. Find the real factors of $F(x)$ in the Examples from 1 to 3.

5. Prove that, if $a + \sqrt{b}$ is a root of $F(x) = 0$, $a - \sqrt{b}$ is a root also. (See proof in § 388.)

Transform the following equations into others whose coefficients shall be whole numbers, that of x^n being unity:

6. $x^3 + \frac{3}{5} x - \frac{7}{5} = 0.$

7. $x^3 - \frac{1}{3} x^2 - \frac{1}{36} x + \frac{1}{108} = 0.$

8. $x^3 - \frac{3}{2} x^2 + \frac{5}{4} x - \frac{2}{3} = 0.$

9. $x^4 - \frac{1}{5} x^3 - \frac{3}{2} x^2 + 0.1\, x + \frac{1}{1250} = 0.$

392. A **Commensurable** real root is one that can be exactly expressed as a whole number or a rational fraction.

An **Incommensurable** real root is one whose exact expression involves surds.

Thus, of the equation $(x-5)(x-\frac{1}{3})(x-\sqrt{2})(x+\sqrt{2})=0$, 5 and $\frac{1}{3}$ are commensurable roots, and $\sqrt{2}$ and $-\sqrt{2}$ incommensurable.

393. Integral Roots. *If the coefficients of* F (x) *are all whole numbers, any commensurable real root of* $F(x)=0$ *is a whole number and an exact division of* p_n.

Suppose $\frac{s}{t}$, a rational fraction in its lowest terms, to be a root of $F(x)=0$; then, by substitution, we have

$$\frac{s^n}{t^n}+p_1\frac{s^{n-1}}{t^{n-1}}+p_2\frac{s^{n-2}}{t^{n-2}}+\cdots+p_n\equiv 0. \qquad (1)$$

Multiplying by t^{n-1}, and transposing, we have

$$\frac{s^n}{t}\equiv -(p_1 s^{n-1}+p_2 t s^{n-2}+\cdots+p_n t^{n-1}). \qquad (2)$$

Now (2) is impossible, for its first member is a fraction in its lowest terms, and its second member is a whole number.

Hence, as a rational fraction cannot be a root, any commensurable root must be a whole number.

Next, let a be an integral root of $F(x)=0$.

Substituting a for x, transposing p_n, and dividing by a, we have

$$a^{n-1}+p_1 a^{n-2}+p_2 a^{n-3}+\cdots+p_{n-1}\equiv -(p_n\div a). \qquad (3)$$

The first member of (3) is integral; hence the quotient $p_n \div a$ is a whole number.

Thus, any commensurable root of $x^3 - 6x^2 + 10x - 8 = 0$ must be ± 1, ± 2, ± 4, or ± 8; for these are the only exact divisors of -8.

394. The **Limits of the Roots** of an equation are any two numbers between which the roots lie.

The limits of the real roots may be found as follows:

Superior Limit. In evaluating $F(6)$ in Example 1 of § 381, the sums are all positive, and they evidently would all be greater for $x > 6$. Hence $F(x)$ can vanish only for $x < 6$; and therefore all the real roots of $F(x) = 0$ are less than 6.

Hence, *if in computing the value of* $\mathrm{F}(\mathrm{c})$, c *being positive, all the sums are positive, the real roots of* $\mathrm{F}(\mathrm{x}) = 0$ *are all less than* c.

Inferior Limit. In evaluating $F(-6)$ in Example 2 of § 381, the sums are alternately $-$ and $+$, and they evidently would all be greater numerically for $x < -6$. Therefore all the real roots of $F(x) = 0$ are greater than -6.

Hence, *if in computing the value of* $\mathrm{F}(\mathrm{b})$, b *being negative, the sums are alternately* $-$ *and* $+$, *all the real roots of* $\mathrm{F}(\mathrm{x}) = 0$ *are greater than* b.

Therefore, if its coefficients are alternately $+$ and $-$, $F(x) = 0$ cannot have any negative roots.

Example. Solve $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$.

In evaluating $F(4)$, the sums are all $+$; and in evaluating $F(-5)$, the sums are alternately $-$ and $+$; hence the real roots of $F(x) = 0$ lie between -5 and 4.

By § 393, the commensurable roots are integral factors of 24. Hence any commensurable root must be ± 1, ± 2, ± 3, or -4.

The work of determining which of these numbers are roots may be arranged as below:

$$\begin{array}{rrrrrl}
1 & +2 & -13 & -14 & +24 & \underline{|1} \\
 & +1 & +3 & -10 & -24 & \\
\hline
1 & +3 & -10 & -24 & \underline{|-2} & \\
 & -2 & -2 & +24 & & \\
\hline
 & +1 & -12 & 0 & &
\end{array}$$

Hence $F(x)$ is divisible by $x - 1$, the quotient $x^3 + 3x^2 - 10x - 24$ is divisible by $x + 2$, and the depressed equation is

$$x^2 + x - 12 = 0,$$

of which the roots are evidently 3 and -4.

Therefore the required roots are $1, -2, 3, -4$.

EXERCISE 55.

1. Show that the real roots of $x^3 - 2x - 50 = 0$ lie between -2 and 4.

2. Show that any commensurable real root of $x^4 - 3x^2 - 75x - 10000 = 0$ is ± 1, ± 2, ± 4, ± 5, ± 8, or 10.

3. Show that the real roots of $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$ lie between -3 and 9, and that any commensurable real root must be ± 1 or 3.

4. Any commensurable root of $x^4 - 15x^2 + 10x + 24 = 0$ must be one of what numbers?

5. Find the roots of the equation in Example 4.

Solve each of the following equations, and verify the roots of each by § 387:

6. $x^5 - 4x^4 - 16x^3 + 112x^2 - 208x + 128 = 0.$

7. $x^4 - 4x^3 - 8x + 32 = 0.$

8. $x^3 - 3x^2 + x + 2 = 0.$

9. $x^3 - 6x^2 + 11x - 6 = 0.$

10. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$

11. $x^3 - 6x^2 + 10x - 8 = 0.$

12. $x^4 - 6x^3 + 24x - 16 = 0.$

13. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$

14. $x^4 - x^3 - 39x^2 + 24x + 180 = 0.$

15. $x^3 + 5x^2 - 9x - 45 = 0.$

16. $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0.$

17. $x^7 + x^6 - 14x^5 - 14x^4 + 49x^3 + 49x^2 - 36x - 36 = 0.$

18. $x^6 + 5x^5 - 81x^4 - 85x^3 + 964x^2 + 780x - 1584 = 0.$

19. $x^3 - 8x^2 + 13x - 6 = 0.$

20. $x^3 + 2x^2 - 23x - 60 = 0.$

21. $x^4 - 45x^2 - 40x + 84 = 0.$

22. $x^6 - 7x^5 + 11x^4 - 7x^3 + 14x^2 - 28x + 40 = 0.$

Solve the following equations by first transforming them into others whose commensurable roots are whole numbers:

23. $x^3 - \frac{4}{6}x^2 - \frac{11}{36}x + \frac{5}{36} = 0.$

24. $8x^3 - 2x^2 - 4x + 1 = 0.$

25. $x^4 - \frac{1}{6}x^3 - \frac{1}{12}x^2 - \frac{13}{24}x + \frac{6}{24} = 0.$

26. $9x^4 - 9x^3 + 5x^2 - 3x + \frac{2}{3} = 0.$

27. $8x^3 - 26x^2 + 11x + 10 = 0.$

28. $x^4 - 6x^3 + 9\frac{1}{2}x^2 - 3x + 4\frac{1}{2} = 0.$

395. Equal Roots. Suppose the equation $F(x) = 0$ has r roots equal to a, and let

$$F(x) \equiv (x-a)^r \phi(x); \qquad (1)$$

then $F'(x) \equiv r(x-a)^{r-1}\phi(x) + (x-a)^r\phi'(x).$ (2)

From (1) and (2) it is evident that $(x-a)^{r-1}$ is a common factor of $F(x)$ and $F'(x)$.

Hence if $F(x) = 0$ has r roots equal to a, $(x-a)^{r-1}$ will be a factor of the H.C.D. of $F(x)$ and $F'(x)$. Any linear factor will occur once more in $F(x)$ than in the H.C.D. of $F(x)$ and $F'(x)$.

EXAMPLE 1. Solve $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0.$(1)

Here $\quad F(x) \equiv x^4 - 11x^3 + 44x^2 - 76x + 48;$

$\therefore \quad F'(x) \equiv 4x^3 - 33x^2 + 88x - 76.$

By the method of § 94 we find the H.C.D. of $F(x)$ and $F'(x)$ to be $x - 2$; hence two roots of (1) are 2 each.

By § 387, the sum of the other two roots is 7, and their product 12; hence the other two roots are 4 and 3.

EXAMPLE 2. Solve

$$x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0. \quad (1)$$

Here the H. C. D. of $F(x)$ and $F'(x)$ is

$$x^4 + 3x^3 + x^2 - 3x - 2. \quad (2)$$

The H. C. D. of function (2) and *its* derivative is $x + 1$; hence $(x + 1)^2$ is a factor of (2). By factoring we obtain

$$x^4 + 3x^3 + x^2 - 3x - 2 \equiv (x + 1)^2 (x + 2)(x - 1). \quad (3)$$

Hence three roots of (1) are -1 each, two -2 each, and two 1 each.

EXERCISE 56.

Solve the following equations, each having equal roots:

1. $x^4 - 14x^3 + 61x^2 - 84x + 36 = 0.$

2. $x^3 - 7x^2 + 16x - 12 = 0.$

3. $x^4 - 24x^2 + 64x - 48 = 0.$

4. $x^4 - 11x^2 + 18x - 8 = 0.$

5. $x^4 + 13x^3 + 33x^2 + 31x + 10 = 0.$

6. $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0.$

7. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$

8. $x^6 + 3x^5 - 6x^4 - 6x^3 + 9x^2 + 3x - 4 = 0.$

9. Show that the equation $x^3 + 3Hx + G = 0$ will have two equal roots, when $4H^3 + G^2 = 0$.

10. If $4r = p^2$, $x^4 - px^2 + r = 0$ will have 3 equal roots.

396. If only two of the roots of a higher numerical equation are incommensurable or imaginary, the commensurable real roots may be found by the methods already given, and the equation depressed to a quadratic, from which the other two roots are readily obtained.

When a higher numerical equation contains no commensurable real root, or when the depressed equation is above the second degree, the following principle is useful in determining the number and situation of the real roots.

397. Change of Sign of $F(x)$. *If* F (b) *and* F (c) *have unlike signs, an odd number of roots of* F (x) = 0 *lies between* b *and* c.

If x changes continuously, then $F(x)$ will pass from one value to another by passing through all intermediate values (§ 255). Therefore to change its sign, $F(x)$ must pass through zero; * for zero lies between any two numbers of opposite signs.

Hence if $F(b)$ and $F(c)$ have opposite signs, $F(x)$ must vanish, or equal 0, for one value, or an odd number of values, of x between b and c.

If $F(b)$ and $F(c)$ have like signs, then we know simply that either no root, or an even number of roots, of $F(x) = 0$ lies between b and c.

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* A function may change its sign by passing through infinity (§ 254); but evidently $F(x)$ or any other integral function of x cannot become infinite for a finite value of x.

Example. Find the situation of the real roots of

$$x^3 - 4x^2 - 6x + 8 = 0.$$

By § 394 we find that all the real roots lie between -2 and 6.

Here $F(-2) = -4, \quad F(0) = +8, \quad F(4) = -18,$

$$F(-1) = +9, \quad F(1) = -1, \quad F(5) = +3.$$

Since $F(-2)$ and $F(-1)$ have unlike signs, at least one root of $F(x) = 0$ lies between -2 and -1. For like reason a second root lies between 0 and 1, and a third between 4 and 5. Hence the roots are $-(1.+)$, $0.+$, and $4.+$.

398. *Every equation of an odd degree has at least one real root whose sign is opposite to that of the known term* p_n.

If $F(x)$ is of an odd degree, then

$$F(-\infty) = -\infty, \; F(0) = p_n, \; F(+\infty) = +\infty.$$

Hence, if p_n is positive, one root of $F(x) = 0$ lies between 0 and $-\infty$ (§ 397); and if p_n is negative, one root lies between 0 and $+\infty$.

399. *Every equation of an even degree in which* p_n *is negative has at least one positive and one negative real root.*

Here $F(-\infty) = +\infty$, $F(0)$ is $-$, $F(+\infty) = +\infty$.

Hence one root of $F(x) = 0$ lies between 0 and $-\infty$, and another between 0 and $+\infty$.

EXERCISE 57.

Find the first figure of each real root of

1. $x^3+x^2-2x-1=0$.
2. $x^3-3x^2-4x+11=0$.
3. $x^4-4x^3-3x+23=0$.
4. $x^3-2x-5=0$.
5. $20x^3-24x^2+3=0$.
6. $x^3-4x^2-6x=-8$.
7. $x^4-4x^3-3x=-27$.
8. $x^3+x-500=0$.
9. $x^3+10x^2+5x=260$.
10. $x^3+3x^2+5x=178$.
11. $x^4-11727x+40385=0$.

STURM'S THEOREM.

*400. The object of Sturm's theorem is to determine the number and situation of the real roots of any numerical equation. Though perfect in theory, Sturm's theorem is laborious in its application. Hence, when possible, the situation of roots is more usually determined by the method of § 397.

*401. **Sturm's Functions.** Let $F(x)=0$ be any equation from which the equal roots have been removed, and let $F'(x)$ denote the first derivative of $F(x)$. Treat $F(x)$ and $F'(x)$ as in finding their H. C. D., with this modification, that the sign of each remainder be changed before it is used as a divisor,

and that no other change of sign be allowed. Continue the operation until a remainder is obtained which does not contain x, and change the sign of that also. Let $F_1(x)$, $F_2(x)$, $\cdots$, $F_m(x^0)$ denote the several remainders with their signs changed; then $F(x)$, $F'(x)$, $F_1(x)$, $F_2(x)$, $\cdots$, $F_m(x^0)$ are called *Sturm's Functions*.

$F(x)$ is the *primitive* function, and $F'(x)$, $F_1(x)$, $\cdots$, $F_m(x^0)$ are the *auxiliary* functions. We use x^0 instead of x in $F_m(x^0)$, since $F_m(x^0)$ does not contain x.

EXAMPLE. Given $x^3 - 3x^2 - 4x + 13 = 0$; find Sturm's functions.

Here $$F(x) \equiv x^3 - 3x^2 - 4x + 13;$$

$$\therefore \quad F'(x) \equiv 3x^2 - 6x - 4.$$

Dividing $F(x)$ by $F'(x)$, first multiplying the former by 3 to avoid fractions, we find that the first remainder of a lower degree than the divisor is $-14x + 35$. Changing the sign of this remainder and rejecting the positive factor 7, we have

$$F_1(x) \equiv 2x - 5.$$

Proceeding in like manner with $3x^2 - 6x - 4$ and $2x - 5$, we find the next remainder to be -1; hence $F_2(x^0) \equiv +1$.

If an equation has equal roots, the process of finding Sturm's functions will discover them, and then we can proceed with the depressed equation.

*402. A **Variation** of sign is said to occur when two successive terms of a series have unlike signs; and a **Permanence**, when they have like signs.

Thus, if the signs of a series of quantities are $+ + - - + + - +$, there are four variations and three permanences of sign.

Again, in the Example of § 401, we have

$$F(x) \equiv x^3 - 3x^2 - 4x + 13, \qquad F_1(x) \equiv 2x - 5,$$

$$F'(x) \equiv 3x^2 - 6x - 4, \qquad F_2(x^0) \equiv +1.$$

When	$F(x)$	$F'(x)$	$F_1(x)$	$F_2(x^0)$	
$x = 0$	$+$	$-$	$-$	$+$	2 variations.
$x = 3$	$+$	$+$	$+$	$+$	0 variations.

* **403. Sturm's Theorem.** *If* F (x) = 0 *has no equal roots and* b *be substituted for* x *in Sturm's functions, and the number of variations noted, and then a greater number* c *be substituted for* x *and the number of variations noted; the first number of variations less the second equals the number of real roots of* F (x) = 0 *that lie between* b *and* c.

(i.) Since each of Sturm's functions is an integral function, to change its sign a Sturmian function must vanish (§ 255).

(ii.) No two of Sturm's functions can vanish for the same value of x.

Since $F(x) = 0$ has no equal roots, $F(x)$ and $F'(x)$ have no common divisor (§ 395). Hence, from the method of finding Sturm's functions, it is evident that no two of them can have a common factor of the form $x - a$, and thus vanish for the same value of x.

(iii.) When any auxiliary function vanishes, the two adjacent functions have opposite signs.

Let the several quotients obtained in the process of finding Sturm's functions be represented by q_1, q_2, q_3, ...; then by principles of division we have

$$F(x) \equiv F'(x)\, q_1 - F_1(x),$$
$$F'(x) \equiv F_1(x)\, q_2 - F_2(x),$$
$$F_1(x) \equiv F_2(x)\, q_3 - F_3(x),$$
$$\cdots \qquad \cdots \qquad \cdots$$

Hence, if any auxiliary function, as $F_2(x)$, vanishes when $x = a$, from the third identity we have

$$F_1(a) \equiv -F_3(a).$$

(iv.) The number of variations of sign of Sturm's functions is not affected by a change of sign of any of the *auxiliary* functions.

Suppose $F_2(x)$ to change its sign when $x = a$; then, by (i.) and (ii.), no other of Sturm's functions can change its sign when $x = a$. Hence all the other functions will have the same sign immediately after $x = a$ that they had immediately before, those of $F_1(x)$ and $F_3(x)$, by (iii.), being *unlike.*

Now, whichever sign be put between two unlike signs, there is one and only one variation. Hence the change of sign of $F_2(x)$ does not affect the *number* of variations of sign. The same holds true of any other auxiliary function except $F_m(x^0)$, which, being constant, cannot change its sign.

(v.) If x increases, there is a loss of one, and only one, variation of sign of Sturm's functions when $F(x)$ vanishes.

When $F(x)$ vanishes, $F'(x)$ is + or −.

If $F'(x)$ is +, $F(x)$ by § 238 is increasing when it vanishes, and therefore must change its sign from − to +. Hence, immediately before $F(x)$ vanishes, we have the variation − +, and immediately afterward the permanence + +.

If $F'(x)$ is −, $F(x)$ is decreasing when it vanishes, and therefore must change its sign from + to −. Hence, when $F(x)$ vanishes, the variation + − becomes the permanence − −.

Whence there is a loss of one variation of Sturm's functions when $F(x)$ vanishes, and only then.

Therefore the number of variations lost while x increases from b to c is equal to the number of roots of $F(x) = 0$ that lie between b and c.

EXAMPLE 1. Determine the number and situation of the real roots of the equation $x^3 - 3x^2 - 4x + 13 = 0$.

By § 394, all the real roots lie between -3 and 4.

Here $F(x) \equiv x^3 - 3x^2 - 4x + 13$, $F_1(x) \equiv 2x - 5$,

$F'(x) \equiv 3x^2 - 6x - 4$, $F_2(x^0) \equiv +1$.

Beginning at $x = -3$, we find the following table of results:

When	$F(x)$	$F'(x)$	$F_1(x)$	$F_2(x)$	
$x = -3$	−	+	−	+	3 variations.
$x = -2$	+	+	−	+	2 variations.
$x = 2$	+	−	−	+	2 variations.
$x = 3$	+	+	+	+	0 variations.

Hence there is one negative root between -3 and -2 (§ 403), and two positive roots between 2 and 3. To separate the two positive roots, we substitute in the Sturmian functions some

value of x between 2 and 3, as 2.5. When $x = 2.5$, the succession of signs is $- - 0 +$, which gives but one variation, whether 0 has the sign + or −; hence one positive root lies between 2 and 2.5, and the other between 2.5 and 3.

EXAMPLE 2. Find the number and situation of the real roots of $2x^4 - 13x^2 + 10x - 19 = 0$.

Sturm's theorem may be applied to an equation in this form, since there is nothing in its demonstration that requires the coefficient of x^n to be unity.

By § 394, the real roots lie between -4 and $+3$.

Here
$$F(x) \equiv 2x^4 - 13x^2 + 10x - 19,$$
$$F'(x) \quad 2(4x^3 - 13x + 5),$$
$$F_1(x) \equiv 13x^2 - 15x + 38.$$

Since, by § 148, the roots of $F_1(x) \equiv 13x^2 - 15x + 38 = 0$ are imaginary, $F_1(x)$ cannot change its sign for any real value of x; hence there can be no loss of variations beyond $F_1(x)$, and it is unnecessary to obtain $F_2(x)$ and $F_3(x^0)$.

When	$F(x)$	$F'(x)$	$F_1(x)$	
$x = -4$	+	−	+	2 variations.
$x = -3$	+	−	+	2 variations.
$x = -2$	−	−	+	1 variation.
$x = 2$	−	+	+	1 variation.
$x = 3$	+	+	+	0 variations.

Hence there are two real roots, one of which is $-(2.+)$ and the other $2.+$.

EXAMPLE 3. Find the number of the real roots of the cubic
$$x^3 + 3Hx + G = 0. \qquad (1)$$

Here $F(x) \equiv x^3 + 3Hx + G$, $F_1(x) \equiv -2Hx - G$,
$$F'(x) \equiv 3(x^2 + H), \qquad F_2(x) \equiv -(G^2 + 4H^3).$$

If $G^2 + 4H^3 > 0$, H may be either + or −; so that

When	$F(x)$	$F'(x)$	$F_1(x)$	$F_2(x)$	
$x = -\infty$	−	+	$\pm$	−	2 variations.
$x = +\infty$	+	+	$\mp$	−	1 variation.

Hence when $G^2 + 4H^3$ is *positive*, only *one* root is real.

If $G^2 + 4H^3 < 0$, evidently H is −; so that we have

$x = -\infty$	−	+	−	+	3 variations.
$x = +\infty$	+	+	+	+	0 variations.

Hence when $G^2 + 4H^3$ is *negative*, all *three* roots are real.

If $G^2 + 4H^3 = 0$, $F(x)$ and $F'(x)$ have $2Hx + G$ as a C. D.; hence, by § 395, two roots are $-G \div 2H$ each. By § 387 the third root is $G \div H$.

EXERCISE 58.

Find the first figure of each real root of

1. $x^3 + 2x^2 - 3x = 9$.
2. $x^3 - 2x - 5 = 0$.
3. $x^3 - 5x^2 + 8x = 1$.
4. $x^3 - x^2 - 2x = -1$.
5. $x^4 - 4x^3 - 3x + 23 = 0$.
6. $x^4 + 4x^3 - 4x^2 - 11x + 4 = 0$.
7. $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$.
8. $x^5 - 10x^3 + 6x + 1 = 0$.

9. Show that in general there are $n + 1$ Sturmian functions.

10. Show that all the roots of $F(x) = 0$ are real when the first term of each of the $n + 1$ Sturmian functions has a positive coefficient.

TRANSFORMATION OF EQUATIONS.

404. In solving an equation it is often advantageous to transform it into another whose roots shall have some known relation to those of the given equation. For one case of transformation see § 391.

405. *To transform an equation into another whose roots shall be those of the first with their signs changed.*

If in $F(x) = 0$, we put $x = -x_1$, we obtain

$$F(-x_1) \equiv (-x_1)^n + p_1(-x_1)^{n-1} + \cdots$$
$$+ p_{n-1}(-x_1) + p_n = 0. \quad (1)$$

Since $x = -x_1$, the roots of $F(x) = 0$ and $F(-x_1) = 0$ are numerically equal with opposite signs.

If in (1) we perform the indicated operations, the terms will be alternately $+$ and $-$ or $-$ and $+$.

Hence, to effect the required transformation, *change the signs of all the terms containing the odd powers of* x, *or of those containing the even powers.*

Thus, the roots of $x^4 - x^2 + 3x + 6 = 0$ are numerically equal to the roots of $x^4 - x^2 - 3x + 6 = 0$, but opposite in sign. The same is true of the roots of $x^5 - 7x^4 + x^3 + 1 = 0$ and $x^5 + 7x^4 + x^3 - 1 = 0$.

406. *To transform an equation into another whose roots shall be the reciprocals of those of the first.*

If in $F(x)=0$ we put $x = 1 \div x_1$, we obtain

$$F\left(\frac{1}{x_1}\right) \equiv \left(\frac{1}{x_1}\right)^n + p_1\left(\frac{1}{x_1}\right)^{n-1} + \cdots + p_{n-1}\frac{1}{x_1} + p_n = 0, \quad (1)$$

which is evidently the equation required.

Multiplying (1) by x_1^n, and reversing the order of the terms, we obtain

$$p_n x_1^n + p_{n-1}x_1^{n-1} + p_{n-2}x_1^{n-2} + \cdots$$
$$+ p_2 x_1^2 + p_1 x_1 + 1 = 0. \quad (2)$$

Hence, to effect the required transformation, *write the coefficients in the reverse order.*

Thus, the roots of $2x^3 - 3x^2 - 4x + 5 = 0$ are the reciprocals of the roots of $5x^3 - 4x^2 - 3x + 2 = 0$.

407. Infinite Roots. If $p_n = 0$, one root of $F(x) = 0$ is 0, and therefore, by § 406, the corresponding root of $F(1 \div x_1) = 0$ is $1 \div 0$, or infinity.

That is, *if in an equation the coefficient of* x^n *is* 0, *one root is infinity.*

Similarly, *if the coefficients of* x^n *and* x^{n-1} *are both* 0, *two roots are infinity; and so on.*

Thus in the linear equation $ax = b$, if $a \doteq 0$, the root $b \div a = \infty$; if $a = 0$, the root $b \div a = b \div 0$, or infinity.

Again, the roots, in § 145, of the quadratic equation

$$ax^2 + bx + c = 0,$$

by rationalizing the numerators, may be put in the forms,

$$\alpha = \frac{2c}{-b-\sqrt{b^2-4ac}}, \quad \beta = \frac{2c}{-b+\sqrt{b^2-4ac}}.$$

Now if $a \doteq 0$, then $\alpha \doteq c \div b$ and $\beta = \infty$; if $a = 0$, one root is finite, and the other is *infinity*.

If $a = 0$ and $b \doteq 0$ also, then $\alpha = \infty$ and $\beta = \infty$; if $a = b = 0$, both roots are *infinity*.

408. *To transform an equation into another whose roots shall be those of the first diminished by a given quantity.*

If in the equation

$$F(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots \\ + p_{n-1} x + p_n = 0, \qquad (1)$$

we put $x = x_1 + h$, we obtain

$$F(x_1 + h) \equiv (x_1 + h)^n + p_1 (x_1 + h)^{n-1} + \cdots \\ + p_{n-1} (x_1 + h) + p_n = 0. \qquad (2)$$

As $x = x_1 + h$, or $x_1 = x - h$, the roots of (2) equal those of (1) diminished by h, h being either positive or negative.

Hence, to effect the required transformation, *substitute* $x_1 + h$ *for* x, *expand, and reduce to the form of* $F(x) = 0$.

409. Computation of the Coefficients of $F(x_1 + h)$. Since $F(x_1 + h)$ may be reduced to the form of $F(x)$, put

$$F(x_1 + h) \equiv x_1^n + q_1 x_1^{n-1} + q_2 x_1^{n-2} + \cdots + q_{n-1} x_1 + q_n.$$

Substituting $x - h$ for x_1, we obtain

$$F(x) \equiv (x - h)^n + q_1 (x - h)^{n-1} + \cdots + q_{n-1} (x - h) + q_n.$$

Dividing $F(x)$ by $x - h$, we obtain

$$(x - h)^{n-1} + q_1(x - h)^{n-2} + \cdots + q_{n-1} \qquad (1)$$

as the quotient, and q_n as the remainder.

Dividing the quotient (1) by $x - h$, we obtain

$$(x - h)^{n-2} + q_1(x - h)^{n-3} + \cdots + q_{n-2}$$

as the quotient, and q_{n-1} as the remainder.

The next remainder will be q_{n-2}; the next q_{n-3}; and so on to q_1.

The last quotient will be the coefficient of x^n.

Hence, *if* F (x) *be divided successively by* x − h, *the successive remainders and the last quotient will be the coefficients of* F (x₁ + h) *in reverse order.*

EXAMPLE. Transform the equation $x^3 - 3x^2 - 2x + 5 = 0$ into another whose roots shall be less by 3.

The work of dividing $F(x)$ successively by $x - 3$ to compute the coefficients of $F(x_1 + 3)$ may be arranged as below:

$$\begin{array}{cccc|l}
1 & -3 & -2 & +5 & \underline{3} \\
 & +3 & +0 & -6 & \\
\hline
1 & 0 & -2 & -1 = q_3 & \\
 & +3 & +9 & & \\
\hline
1 & +3 & +7 = q_2 & & \\
 & +3 & & & \\
\hline
1 & +6 = q_1 & & &
\end{array}$$

Hence the transformed equation is

$$F(x_1 + 3) = x_1^3 + 6x_1^2 + 7x_1 - 1 = 0.$$

410. Equation Lacking any Term. If the binomials in equation (2) of § 408 be expanded, the coefficient of x_1^{n-1} will evidently be $nh + p_1$; hence if we put $nh + p_1 = 0$, or $h = -p_1 \div n$, the transformed equation will lack the term in x_1^{n-1}.

In like manner an equation can be transformed into another which shall lack any specified term.

EXAMPLE. Transform the equation $x^3 - 6x^2 + 4x + 5 = 0$ into another lacking the term in x^2.

Here $$p_1 = -6,\ n = 3;$$

$$\therefore\ h = -p_1 \div n = 2.$$

Transforming the given equation into another of which the roots are less by 2, and writing x for x_1, we obtain

$$x^3 - 8x - 3 = 0,$$

an equation which lacks the term in x^2.

EXERCISE 59.

Transform each of the following equations into another having the same roots with opposite signs:

1. $x^6 + x^5 - x^2 - 5x + 7 = 0.$

2. $x^4 - 7x^3 - 5x^2 + 8 = 0.$

3. $x^5 - 6x^4 - 7x^2 + 5x = 3.$

4. $x^7 - 7x^6 + 4x^2 - 7x + 2 = 0.$

Find the equation whose roots are the reciprocals of those of each of the following equations:

5. $x^3 - 7x^2 - 4x + 2 = 0.$ 7. $x^5 - x^3 + 5x^2 + 8 = 0.$

6. $x^4 - 8x^2 + 7 = 0.$ 8. $x^6 - x^4 + 7x^3 + 9 = 0.$

Transform each of the following equations into another whose roots shall be less by the number placed opposite the equation:

9. $x^3 - 3x^2 - 6 = 0.$ 5.

10. $x^5 - 2x^4 + 3x^2 + 4x - 7 = 0.$ 4.

11. $x^4 - 2x^3 + 3x^2 + 5x + 7 = 0.$ $-2.$

12. $x^4 - 18x^3 - 32x^2 + 17x + 19 = 0.$ 5.

13. $5x^4 + 28x^3 + 51x^2 + 32x - 1 = 0.$ $-2.$

Transform each of the following equations into another which shall lack the term in x^2:

14. $x^3 - 3x^2 + 3x + 4 = 0.$

15. $x^3 - 6x^2 + 8x - 2 = 0.$

16. $x^3 + 6x^2 - 7x - 2 = 0.$

17. $x^3 - 9x^2 + 12x + 19 = 0.$

411. Horner's Method of Solving Numerical Equations. By this method any real root is obtained, after its situation has been determined. The main

principle involved is the successive diminution of the roots of the given equation by known quantities, as explained in § 408.

Thus, suppose that one root of $F(x) = 0$ is found to lie between 40 and 50; to find this root we transform the equation $F(x) = 0$ into another whose roots shall be less by 40, and obtain

$$F(40 + x_1) = 0, \quad (1)$$

of which the positive root sought is less than 10.

If this root is found to lie between 6 and 7, we transform equation (1) into another whose roots shall be less by 6, and obtain

$$F[40 + (6 + x_2)] \equiv F(46 + x_2) = 0, \quad (2)$$

of which the positive root sought is less than 1.

If this root lies between 0.5 and 0.6, we transform equation (2) into another whose roots shall be less by 0.5, and obtain

$$F(46.5 + x_3) = 0, \quad (3)$$

of which the positive root sought is less than 0.1.

First, suppose this root of (3) to be 0.03, we then have $F(46.53) = 0$; that is, one root of $F(x) = 0$ is 46.53.

Next, suppose this root of (3) to lie between 0.03 and 0.04; then it follows that one root of $F(x) = 0$ lies between 46.53 and 46.54.

By transforming equation (3) into another whose roots shall be less by 0.03, the thousandths figure of the root can be found; and so on.

By repeating these transformations we can evidently obtain a root exactly, or may approximate to any root as nearly as we please.

412. One of the *practical advantages* of Horner's method is that the first figure of the root of any

transformed equation, after the first, is in general correctly obtained by dividing the last coefficient with its sign changed by the preceding coefficient, which is therefore often called the *trial divisor.*

The figure obtained in this way from the first transformed equation is likely to be too large.

Example. Find the root of the equation

$$F(x) \equiv x^3 - 3x^2 - 4x + 11 = 0 \qquad \text{(a)}$$

that lies between 3 and 4.

We first transform equation (a) into another whose roots shall be less by 3. The work is given below.

$$\begin{array}{cccc|l}
1 & -3 & -4 & +11 & \underline{3} \\
 & +3 & +0 & -12 & \\
\hline
 & 0 & -4 & -1 & \\
 & +3 & +9 & & \\
\cline{2-3}
 & +3 & +5 & & \\
 & +3 & & & \\
\cline{2-2}
 & +6 & & &
\end{array}$$

Thus $F(3) \equiv -1$, and the first transformed equation is

$$F(3 + x_1) \equiv x_1^3 + 6x_1^2 + 5x_1 - 1 = 0, \qquad (1)$$

of which the root sought is positive and less than 1.

Since $x_1 < 1$, $x_1^3 < x_1^2 < x_1$. Hence it is probable that the *first* figure of this root of (1) will be correctly given by the quadratic equation

$$6x_1^2 + 5x_1 - 1 = 0; \quad \therefore\ x_1 = 1 = 0.1 +.$$

We next diminish the roots of (1) by 0.1.

1	+ 6.0	+ 5.00	− 1.000	0.1
	+ 0.1	+ 0.61	+ 0.561	
	+ 6.1	+ 5.61	− **0.439**	
	+ 0.1	+ 0.62		
	+ 6.2	+ **6.23**		
	+ 0.1			
	+ **6.3**			

Thus $F(3.1) \equiv -0.439$, and the second transformed equation is

$$F(3.1 + x_2) \equiv x_2^3 + 6.3\,x_2^2 + 6.23\,x_2 - 0.439 = 0, \qquad (2)$$

of which the root sought is positive and less than 0.1.

Since $x_2 < 0.1$, x_2^3 and x_2^2 are each much smaller than x_2. It is probable therefore that the *first* figure of this root of (2) will be correctly given by the simple equation

$$6.23\,x_2 - 0.439 = 0; \quad \therefore x_2 = 0.06 +.$$

We next diminish the roots of (2) by 0.06.

1	+ 6.30	+ 6.2300	− 0.439000	0.06
	+ 0.06	+ 0.3816	+ 0.396696	
	+ 6.36	+ 6.6116	− **0.042304**	
	+ 0.06	+ 0.3852		
	+ 6.42	+ **6.9968**		
	+ 0.06			
	+ **6.48**			

Thus $F(3.16)$ is −, and the third transformed equation is

$$F(3.16 + x_3) \equiv x_3^3 + 6.48\,x_3^2 + 6.9968\,x_3 - 0.042304 = 0, \qquad (3)$$

of which the root sought is positive and less than 0.01.

Dividing 0.042304 by 6.9968 we find the next figure of the root to be 0.006. Diminishing the roots of (3) by 0.006 will give the next transformed equation, which will furnish the next figure of the root; and so on.

But, since $x_3 < 0.01$, and the coefficient of x_3^2 is less than that of x_3, it is probable that the first *two* figures of x_3 will be correctly given by the simple equation

$$x_3 = 0.042304 \div 6.9968 = 0.0060+.$$

Hence 3.1660 is the required root of (a) to four places of decimals.

Observe that the known term of any transformed equation is the value of $F(x)$ for the part of the root thus far found.

413. As seen above, the known term of any transformed equation is the value of $F(x)$ for the part of the root thus far found; hence the known term must have the same sign in all the transformed equations. If any figure of the root is taken too large, the known term in the next equation will have the wrong sign. If a figure is taken too small, the root of the next equation will evidently be of the same order of units.

Hence, *each figure of the root is correct if the next transformed equation has a known term of the same sign as that of the preceding equation and a root of a lower order of units.*

EXAMPLE. Find the root of $x^3 - 3x^2 - 4x + 11 = 0$ that lies between 1 and 2.

We give below the work of the successive transformations written together in the usual form. The broken lines mark the

conclusion of each transformation, and the figures in black-letter are the coefficients of the successive transformed equations.

1. -3	-4	$+11$	1.782
1	-2	-6	
-2	-6	**5.000**	
1	-1	-4.557	
-1	**$-$7.00**	**0.443000**	
1	0.49	-0.428448	
0.0	-6.51	**0.014552000**	
0.7	0.98	-0.010340232	
0.7	**$-$5.5300**	**0.004211768**	
0.7	0.1744		
1.4	-5.3556		
0.7	0.1808		
2.10	**$-$5.174800**		
0.08	4684		
2.18	-5.170116		
0.08	4688		
2.26	**$-$5.165428**		
0.08			
2.340			
2			
2.342			
2			
2.344			
2			
2.346			

Here we find that the *second* figure of the root is correctly given by dividing 5 by 7; the third by dividing 0.443 by 5.53; the fourth by dividing 0.014552 by 5.1748; and so on.

Since $x_4 < 0.001$, and the coefficient of x_4^2 is much less than that of x_4, it is probable that the first three figures of x_4 are correctly given by $x_4 = 0.004211768 \div 5.165428 = 0.000815$. Hence 1.782815 is the required root to six places of decimals.

How many figures of the root will in this way be correctly given by the last transformed equation may be inferred from the value of its root and the relative values of its leading coefficients.

414. Negative Roots. To find a negative root, it is simplest to change the sign of the roots (§ 405), obtain the corresponding positive root, and change its sign.

Thus to find the root of $x^3 - 3x^2 - 4x + 11 = 0$ that lies between -1 and -2, we obtain the positive root of the equation

$$x^3 + 3x^2 - 4x - 11 = 0 \qquad \S\ 405.$$

that lies between 1 and 2, and change its sign.

It is evident that Horner's method is directly applicable to an equation in which the coefficient of x^n is not unity.

NOTE. For a fuller discussion of Horner's method, for its application to cases where roots are very nearly equal, and for contractions of the work, see Burnside and Panton's or Todhunter's "Theory of Equations."

EXERCISE 60.

Find to five places of decimals the root of the equation

1. $x^3 + x - 500 = 0$, that is $7 +$.

2. $x^3 - 2x - 5 = 0$, that is $2 +$.

3. $x^3 - 5x^2 + 8x - 1 = 0$, that is $0 +$.

4. $x^3 + 2x^2 - 3x - 9 = 0$, that is $1 +$.

5. $2x^3 + 3x - 90 = 0$, that is $3 +$.

6. $x^3 + x^2 + 70x + 300 = 0$, that is $-(3 +)$.

7. $5x^3 - 6x^2 + 3x + 85 = 0$, that is $-(2 +)$.

8. $3x^4 + x^3 + 4x^2 + 5x - 375 = 0$, that is $3 +$.

9. $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$, that lies between 30 and 40.

Find the positive root of each of the equations:

10. $2x^3 - 85x^2 - 85x - 87 = 0$.

11. $4x^3 - 13x^2 - 31x - 275 = 0$.

12. $20x^3 - 121x^2 - 121x - 141 = 0$.

13. Solve $x^3 - 315x^2 - 19684x + 2977260 = 0$.

Reciprocal Equations.

415. A *Reciprocal*, or *Recurring*, equation is one that remains unaltered when x is changed into its reciprocal; that is, when the coefficients are written in reverse order (§ 406). Hence the reciprocal of any root of a reciprocal equation is also a root.

Therefore, if the equation is of an odd degree, one root is its own reciprocal; hence one root of a reciprocal equation of an odd degree is $+1$ or -1.

EXAMPLE. Given 5 as one root, to solve the equation

$$5x^5 - 51x^4 + 160x^3 - 160x^2 + 51x - 5 = 0. \quad (1)$$

Since (1) is a reciprocal equation, and one root is 5, a second root is $\frac{1}{5}$. Since (1) is of an odd degree, one root is $+1$ or -1. By inspection we see that $+1$ is a root.

The depressed equation is

$$x^2 - 4x + 1 = 0,$$

from which $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$, which equals $\dfrac{1}{2+\sqrt{3}}$.

A reciprocal equation of an odd degree can be depressed to one of an even degree; for one of its roots is always known.

416. Let $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0, \quad (1)$

and $p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_1x + 1 = 0, \quad (2)$

be equivalent equations; that is, let (1) be a reciprocal equation. Dividing (2) by p_n to put it in the form of (1), and then equating their last terms, we have

$$p_n = 1 \div p_n; \quad \therefore \; p_n = \pm 1.$$

Reciprocal equations are divided into two classes, according as p_n is $+1$ or -1.

First Class. If $p_n = 1$; then, from (1) and (2),

$$p_1 = p_{n-1}, \; p_2 = p_{n-2}, \cdots;$$

that is, *the coefficients of terms equidistant from the ends of* F(x) *are equal.*

Second Class. If $p_n = -1$; then, from (1) and (2),

$$p_1 = -p_{n-1},\ p_2 = p_{n-2}, \cdots;$$

that is, *the coefficients of terms equidistant from the ends of* F (x) *are equal numerically but opposite in sign.*

If in this class n is even, say $n = 2m$; then $p_m = -p_m$, or $p_m = 0$; that is, the middle term is wanting.

417. Standard Form. Any equation of the second class of even degree can evidently be written in the form

$$x^n - 1 + p_1 x(x^{n-2} - 1) + \cdots = 0. \qquad (1)$$

Since n is even, $F(x)$ in (1) is divisible by $x^2 - 1$; hence two roots of (1) are ± 1.

The depressed equation will evidently be a reciprocal equation of the first class of even degree, which is called *the standard form of reciprocal equations.*

Hence, *any reciprocal equation is in the standard form or can be depressed to that form.*

418 Any reciprocal equation of the *standard* form can be reduced to one of half its degree. The following example will illustrate this truth.

EXAMPLE. Solve $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$.

Dividing by $(x^4)^{\frac{1}{2}}$, or x^2, we obtain

$$\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0. \qquad (1)$$

Since $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$, from (1) we have

$$\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) = -4;$$

$$\therefore\ x + \frac{1}{x} = 4 \text{ or } 1;$$

$$\therefore\ x = 2 \pm \sqrt{3}, \frac{1 \pm \sqrt{-3}}{2}.$$

EXERCISE 61.

Solve the following equations:

1. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$
2. $2x^4 - 3x^3 - 6x^2 + 3x + 2 = 0.$
3. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$
4. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$
5. $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0.$
6. $x^6 - x^5 + x^4 - x^2 + x - 1 = 0.$
7. $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$
8. $x^3 - bx^2 + bx - 1 = 0.$

419. Binomial Equations. The two general forms of Binomial Equations are

$$x^n - b = 0, \qquad (1)$$

and

$$x^n + b = 0, \qquad (2)$$

in which b is any positive number.

By § 405, *when* n *is odd, the roots of* (2) *are the roots of* (1) *with their signs changed.*

The n roots of either (1) or (2) are unequal; for $x^n \mp b$ and its derivative nx^{n-1} have no C. D.

From (1), $x = \sqrt[n]{b}$; from (2), $x = \sqrt[n]{-b}$; that is, each of the n unequal roots of (1) or (2) is an nth root of $+b$ or $-b$.

Hence, *any number has* n *unequal* n*th roots.*

By § 391, the n roots of $x^n - 1 = 0$ multiplied by $\sqrt[n]{a}$ are equal to the n roots of

$$x^n - a = 0.$$

Hence, *all the* n*th roots of any number may be obtained by multiplying any one of them by the* n*th roots of unity.*

If n is even, $x^n - 1 = 0$ has two real roots, ± 1.

If n is even, $x^n + 1 = 0$ has no real root; for $\sqrt[n]{-1}$ is imaginary when n is even.

If n is odd, $x^n - 1 = 0$ has one real root, $+1$; and $x^n + 1 = 0$ has one real root, -1.

420. The Cube Roots of Unity. The roots of

$$x^3 - 1 \equiv (x - 1)(x^2 + x + 1) = 0,$$

were found to be

$$1, \ -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \ -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

If ω denote either of the imaginary roots, by actually squaring, the other is found to be ω^2. Hence the three cube roots of $+1$ are 1, ω, and ω^2.

Therefore by § 419 the three roots of $x^3 + 1 = 0$, or the three cube roots of -1, are -1, $-\omega$, and $-\omega^2$.

EXAMPLE. Find the five fifth roots of 32 and -32.

The equation $x^5 = 1$ is equivalent to $x - 1 = 0$

and $$x^4 + x^3 + x^2 + x + 1 = 0. \qquad (1)$$

From (1) $$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 1;$$

$$\therefore\ x + \frac{1}{x} = \frac{-1 \pm \sqrt{5}}{2}. \qquad (2)$$

Solving $x - 1 = 0$ and the two quadratics in (2), and multiplying each root by $\sqrt[5]{32}$, or 2, we find the five fifth roots of 32 to be 2,

$$\frac{-1 + \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}}{2},\ \frac{-1 - \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}}{2}.$$

These roots with their signs changed are the roots of -32.

***421. Solution of Cubic Equations.** By § 410 the general cubic equation

$$x^3 + p_1 x^2 + p_2 x + p^3 = 0$$

can be transformed into another of the simpler form

$$x^3 + 3Hx + G = 0. \qquad (1)$$

To solve this equation, assume

$$x = r^{\frac{1}{3}} + s^{\frac{1}{3}}; \qquad (2)$$

$$\therefore\ x^3 = r + s + 3 r^{\frac{1}{3}} s^{\frac{1}{3}} \left(r^{\frac{1}{3}} + s^{\frac{1}{3}}\right);$$

$$\therefore\ x^3 - 3 r^{\frac{1}{3}} s^{\frac{1}{3}} x - (r + s) = 0. \qquad (3)$$

Comparing coefficients in (1) and (3), we have

$$r^{\frac{1}{3}} s^{\frac{1}{3}} = -H, \; r + s = -G. \tag{4}$$

Solving these equations, we obtain

$$r = \tfrac{1}{2}\left(-G + \sqrt{G^2 + 4H^3}\right), \tag{5}$$

$$s = \tfrac{1}{2}\left(-G - \sqrt{G^2 + 4H^3}\right).$$

Substituting in (2) the value of $s^{\frac{1}{3}}$ obtained from the first of equations (4), we have

$$x = r^{\frac{1}{3}} + \frac{-H}{r^{\frac{1}{3}}}, \tag{6}$$

the value of r being given in (5).

If $\sqrt[3]{r}$ denote any one of the cube roots of r, by § 419, $r^{\frac{1}{3}}$ will have the three values, $\sqrt[3]{r}$, $\omega \sqrt[3]{r}$, $\omega^2 \sqrt[3]{r}$, and the three roots of (1) will be

$$\sqrt[3]{r} + \frac{-H}{\sqrt[3]{r}}, \quad \omega \sqrt[3]{r} + \frac{-H}{\omega \sqrt[3]{r}}, \quad \omega^2 \sqrt[3]{r} + \frac{-H}{\omega^2 \sqrt[3]{r}}.$$

If in (6) we replace r by s, the values of x will not be changed; for, by (2) and the relation $r^{\frac{1}{3}} s^{\frac{1}{3}} = -H$, the terms are then simply interchanged. Moreover, the other solution of equations (4) would evidently repeat these values of x.

NOTE. The above solution is generally known as *Cardan's Solution*, as it was first published by him, in 1545. Cardan obtained it from Tartaglia; but it was originally due to Scipio Ferreo, about 1505. See historical note at the end of Burnside and Panton's "Theory of Equations."

*** 422. Application to Numerical Equations.** When a numerical cubic has a pair of *imaginary* or *equal* roots, by Example 3 of § 403, $G^2 + 4H^3 >$ or $= 0$; hence r in (5) of § 421 is real, and therefore the roots may be computed by the formula (6).

When, however, the roots of a cubic are all *real* and *unequal*, by Example 3 of § 403, $G^2 + 4H^3 < 0$; whence r is imaginary, and the roots involve the cube root of a complex number. Hence, as there is no general arithmetical method of extracting the cube root of a complex number, the formula is useless for purposes of arithmetical calculation. In this case, however, the roots may be computed by methods involving Trigonometry.

When the real root of a cubic has been found by (6) or (2) of § 421 it is simpler to find the other two roots from the depressed equation.

EXAMPLE. Solve $x^3 - 15x - 126 = 0$. (1)

Put $$x = r^{\frac{1}{3}} + s^{\frac{1}{3}};$$

$$\therefore \quad x^3 - 3r^{\frac{1}{3}}s^{\frac{1}{3}}x - (r+s) = 0; \qquad (2)$$

$$\therefore \quad r^{\frac{1}{3}}s^{\frac{1}{3}} = 5, \; r + s = 126;$$

$$\therefore \quad r^{\frac{1}{3}} = 5, \; s^{\frac{1}{3}} = 1;$$

$$\therefore \quad x = r^{\frac{1}{3}} + s^{\frac{1}{3}} = 5 + 1 = 6.$$

Hence the depressed equation is

$$x^2 + 6x + 21 = 0,$$

and the three roots are $6, \; -3 + 2\sqrt{-3}, \; -3 - 2\sqrt{-3}$.

EXERCISE 62.

1. Find the 6 sixth roots of 729; of -729.

2. Find the 8 eighth roots of 256; of -256.

Solve the following equations:

3. $x^3 - 18x = 35$.

4. $x^3 + 63x = 316$.

5. $x^3 + 72x = 1720$.

6. $x^3 + 21x = -342$.

7. $x^3 + 3x^2 + 9x = 13$.

8. $x^3 - 6x^2 + 3x = 18$.

9. $x^3 - 6x^2 + 13x = 10$.

10. $x^3 - 15x^2 - 33x = -847$.

* **423. Solution of Biquadratic Equations.** Any biquadratic equation can be put in the form

$$x^4 + 2px^3 + qx^2 + 2rx + s = 0. \qquad (1)$$

Adding $(ax+b)^2$ to both members, we obtain

$$x^4 + 2px^3 + (q + a^2)x^2 + 2(r + ab)x + s + b^2 = (ax + b)^2. \qquad (2)$$

Assume as an identity

$$x^4 + 2px^3 + (q + a^2)x^2 + 2(r + ab)x + s + b^2 \equiv (x^2 + px + k)^2. \qquad (3)$$

Equating coefficients (§ 262), we have

$$p^2 + 2k = q + a^2,\ pk = r + ab,\ k^2 = s + b^2. \qquad (4)$$

Eliminating a and b from (4), we have

$$(p k - r)^2 = (2 k + p^2 - q)(k^2 - s),$$

or $\quad 2 k^3 - q k^2 + 2(p r - s) k + p^2 s - q s - r^2 = 0.$

From this cubic, find a real value of k (§ 389). The values of a and b are then known from (4). Subtracting (2) from (3), we have

$$(x^2 + p x + k)^2 - (a x + b)^2 = 0,$$

which is equivalent to the two quadratics

$$x^2 + (p - a) x + (k - b) = 0,$$

and $$x^2 + (p + a) x + (k + b) = 0,$$

from which the roots of (1) are readily obtained.

NOTE. The solution given above is that of Ferrari, a pupil of Cardan. This and those of Descartes, Simpson, Euler, and others, all depend upon the solution of a cubic by Cardan's method, and will of course fail when that fails. For a full discussion of Reciprocal, Cubic, and Biquadratic Equations consult Burnside and Panton's "Theory of Equations."

EXAMPLE. Solve $x^4 - 6 x^3 + 12 x^2 - 14 x + 3 = 0$.

Adding $(a x + b)^2$ to both members, we obtain

$$x^4 - 6 x^3 + (12 + a^2) x^2 + 2(a b - 7) x + b^2 + 3 = (a x + b)^2. \quad (1)$$

Since $p = -3$, assume as an identity

$$x^4 - 6 x^3 + (12 + a^2) x^2 + 2(a b - 7) x + b^2 + 3 \equiv (x^2 - 3 x + k)^2; \quad (2)$$

$$\therefore\ 12 + a^2 = 9 + 2 k,\ a b - 7 = -3 k,\ b^2 + 3 = k^2;$$

$$\therefore\ (7 - 3 k)^2 = (k^2 - 3)(2 k - 3);$$

$$\therefore\ k^3 - 6 k^2 + 18 k - 20 = 0.$$

Whence $k = 2$; hence $a^2 = 1$, $b^2 = 1$, $ab = 1$. (3)

From (1), (2), and (3), we obtain

$$(x^2 - 3x + 2)^2 - (x + 1)^2 = 0,$$

which is equivalent to the two equations

$$x^2 - 4x + 1 = 0, \quad x^2 - 2x + 3 = 0;$$

$$\therefore \quad x = 2 \pm \sqrt{3}, \quad 1 \pm \sqrt{-2}.$$

EXERCISE 63.

Solve the following equations:

1. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0.$

2. $x^4 - 3x^2 - 42x - 40 = 0.$

3. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$

4. $x^4 - 3x^2 - 6x - 2 = 0.$

5. $x^4 - 14x^3 + 59x^2 - 60x - 36 = 0.$

6. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0.$

7. $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$

THE END.